Session 3
Sep. 10, 2018

some identities:

- $\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$
- $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = (1-1)^n = 0$
- $\sum_{k=0}^{n} \binom{m-1+k}{k} = \binom{m+n}{n} \qquad (m \geq 1)$

  ↳ can be proven by induction (HW)

- HW: prove that $\sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x} \quad (x \neq 1)$

  Later: $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad (|x|<1)$, geometric series

- HW next week: Theorem: $(1-x)^{-m} = \sum_{k=0}^{\infty} \binom{m+k-1}{k} x^k$
  (HW sheet 2)
  $|x|<1$, and $m \geq 1$ (integer)

1.3 Limits

limits are basis of Analysis (real numbers, derivatives, integrals, ...)

sequence: $(a_n)_{n\in\mathbb{N}} = (a_n)_{n\geq 1} = (a_1, a_2, a_3, \ldots)$

here, we usually deal with seq.s of real numbers

Examples: $\cdot\ (a_n)_{n\in\mathbb{N}}$, with $a_n = n$

$\cdot\ (a_n)_{n\in\mathbb{N}}$, with $a_n = \frac{1}{n^2}$, i.e., $(a_n)_{n\geq 1} = \left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \ldots\right)$

central concept: convergence of seq.s

loose definition: If the a_n's come arbitrarily close to some number a for large n, we say $(a_n)_{n\in\mathbb{N}}$ converges to a.
The number a is called the limit of the sequence, and we write $a = \lim\limits_{n\to\infty} a_n$ or $a_n \xrightarrow{n\to\infty} a$.

If no such a exists, the seq. diverges.

for rigorous definition:

- arbitrarily close $\Leftrightarrow$ for all $\varepsilon > 0$ (no matter how small)
- arbitrarily close to a $\Leftrightarrow$ $|a_n - a| < \varepsilon$
- for large n $\Leftrightarrow$ there is some $N\in\mathbb{N}$, large depending on ε, s.t.
  $|a_N - a| < \varepsilon$ (or $|a_n - a| < \varepsilon$ for all $n \geq N$)

<u>Formal/rigorous def. of convergence:</u>

For all $\varepsilon > 0$ $(\varepsilon\in\mathbb{R})$ there is an $N\in\mathbb{N}$, s.t. $|a_n - a| < \varepsilon$ for all $n \geq N$ (for some a).

<u>Ex.:</u> · $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0$

· $\lim\limits_{n\to\infty} \frac{n}{n+1} = \lim\limits_{n\to\infty} \frac{\frac{1}{n}\cdot n}{\frac{1}{n}\cdot n + \frac{1}{n}\cdot 1} = \lim\limits_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$ (need to "pull lim inside") or: $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n+1-1}{n+1} = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1$

- $\lim_{n\to\infty} \frac{2n^2-1}{n^2+1} = 2$

- $a_n = n$ diverges

- $a_n = (-1)^n$ diverges (but not $a_n \to \pm\infty$)

- geometric progression $a_n = q^n$ for some $q \in \mathbb{R}$

$$\lim_{n\to\infty} q^n = \begin{cases} 0 & , \text{ for } |q|<1 \\ 1 & , \text{ for } q=1 \\ \text{diverges to } \infty & \text{for } q>1 \\ \text{diverges} & \text{for } q<-1 \end{cases}$$

<u>Properties</u>: 1) If $a_n \xrightarrow{n\to\infty} a$, $b_n \xrightarrow{n\to\infty} b$, then

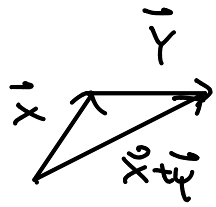
- $a_n + b_n \xrightarrow{n\to\infty} a+b$ $(*)$
- $a_n \cdot b_n \xrightarrow{n\to\infty} a \cdot b$
- $\frac{a_n}{b_n} \xrightarrow{n\to\infty} \frac{a}{b}$ if $b_n \neq 0$ for all n and $b \neq 0$

formal proof of $(*)$: we know $a_n \xrightarrow{n\to\infty} a$, i.e., $|a_n - a|$ arbitrarily small for large n, and same for b_n.

Then $|(a_n+b_n)-(a+b)| = |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b|$
which is arbitrarily small.

note: $|x+y| \leq |x|+|y|$ is called the triangle inequality
(check for real numbers ...)

later: also true for vectors



2) If seq. $(a_n)_{n\in\mathbb{N}}$ converges, it is bounded, i.e., there is some B, s.t.
for all $n\in\mathbb{N}$, we have $|a_n| \leq B$.
Converse is not true, e.g., $(-1)^n$ is bounded but doesn't converge.
Statement in mathematical notation:

$$(a_n)_{n\in\mathbb{N}} \text{ converges} \Longrightarrow \underbrace{\exists}_{\text{there exists}} B>0, \text{ s.t. } \underbrace{\forall}_{\text{forall}} n\in\mathbb{N}: |a_n| \leq B$$

3) Def.: A sequence $(a_n)_{n\in\mathbb{N}}$ is called Cauchy sequence if

$$\forall \varepsilon>0 \ \exists N\in\mathbb{N}, \text{ s.t. } |a_n - a_m| < \varepsilon \quad \forall n,m \geq N.$$

note: for $a_n \in \mathbb{R}$, convergence of $(a_n)_{n\in\mathbb{N}} \Longleftrightarrow (a_n)_{n\in\mathbb{N}}$ Cauchy

extra note:

"$\Longrightarrow$" always true: $|a_n - a_m| = |a_n - a - a_m + a| \underset{\text{triangle ineq.}}{\leq} |a_n - a| + |a_m - a|$

"$\Longleftarrow$" not necessarily true. Consider, e.g., Cauchy sequences of rational numbers. Those might not have a limit in the rational numbers, i.e., they don't converge in the rational numbers, but they could have, e.g., $\sqrt{2}$ as a limit.