Session 4
Sep. 12, 2018

Supremum and infimum:

Let A be a subset of the real numbers, $A \subset \mathbb{R}$, all numbers $a \in \mathbb{R}$ with $a \geq x \ \forall x \in A$ are called upper bound ($a \leq x$: lower bound)

Def.: supremum of $A = \sup A \underset{\sim}{:=}$ smallest upper bound

means: "is defined as"

infimum of $A = \inf A :=$ biggest a, s.t. $a \leq x \ \forall x \in A$
(biggest lower bound)

Ex.:
- $\sup\{0 < x < 2\} = 2 = \sup(0,2)$ (but $2 \notin (0,2)$)
- $\inf(0,2) = 0$
- $\sup\{\frac{1}{n} : n \in \mathbb{N} \ (n \geq 1)\} = 1$
- $\inf\{\frac{1}{n} : n \in \mathbb{N} \ (n \geq 1)\} = 0$

$\big((0,2) = [0,2] \setminus (\{0\} \cup \{2\})\big)$

open interval, closed interval, without, union

Def.:
- $\limsup\limits_{n\to\infty} a_n := \lim\limits_{m\to\infty} \sup\limits_{n \geq m} a_n$
- $\liminf\limits_{n\to\infty} a_n := \lim\limits_{m\to\infty} \inf\limits_{n \geq m} a_n$

Ex.:
- $\limsup\limits_{n\to\infty} (-1)^n = 1$
- $\liminf\limits_{n\to\infty} (-1)^n = -1$

note: limsup and liminf either exist or are $\pm\infty$. So they can exist even if lim doesn't.

1.4 Continuity

more about functions:

$$f: \underset{\text{domain}}{A} \underbrace{\longrightarrow}_{\text{to}} \underset{\text{codomain}}{B}, \quad x \underbrace{\longmapsto}_{\text{maps to}} f(x)$$

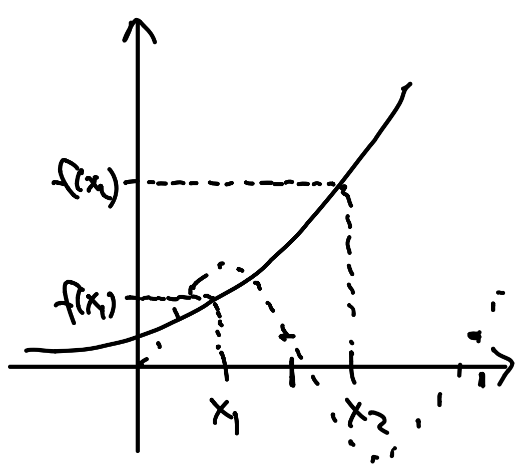
range or image of f is $\text{Im}(f) := \{ f(x) \text{ s.t. } x \in A \}$

Ex.:

- $f: \mathbb{R} \to \mathbb{R}, x \mapsto f(x) = x^2$, the image is $[0, \infty)$
- $f: \mathbb{R} \to [0, \infty), x \mapsto f(x) = x^2$

strictly speaking different fct.s

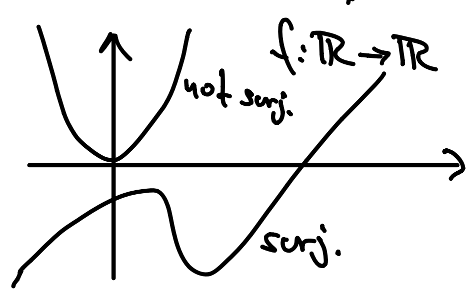
- $f: [0, \infty) \to [0, \infty), x \mapsto f(x) = \sqrt{x}$



Def.:

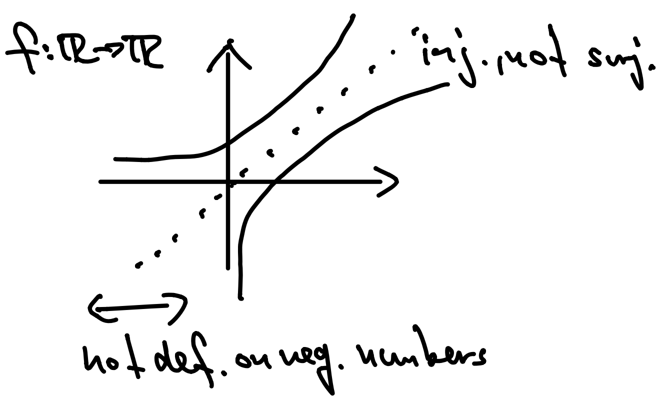
- f is called injective or one-to-one
  $f(x_1) \neq f(x_2) \ \forall x_1 \neq x_2$

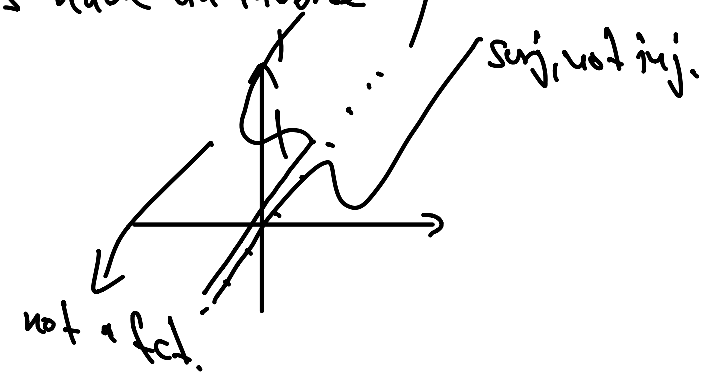
- f is called surjective or onto if
  $\forall y \in B \ \exists x \in A$ with $f(x) = y$
  (im = codomain)



- if f is both injective and surjective, it is called **bijective**.

(relevance: bijective fct.s have an inverse)



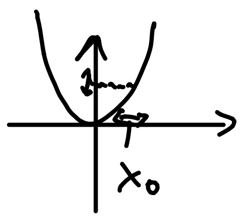


loose def. of continuity:

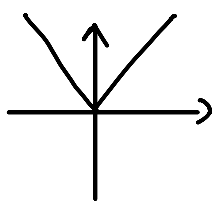
f is cont. at x_0, if arbitrarily small changes in x_0 produce arbitrarily small changes in $f(x)$

note: if f is cont. $\forall x_0 \in$ domain of f, then f is called cont.

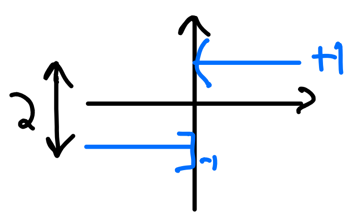
Ex.: $\cdot$ $f(x) = x^2$ is cont.
(and so are all polynomials)



$\cdot$ $f(x) = |x|$ is cont.



$\cdot$ $f(x) = \begin{cases} 1, & \text{for } x > 0 \\ -1, & \text{for } x \leq 0 \end{cases}$ is not cont.



(it is not cont. at 0)

rigorous def. via sequences: $f: A \to \mathbb{R}$ is cont. at $x_0 \in A$ means:

$\forall (a_n)_{n \in \mathbb{N}}$ with $a_n \in A$ and $a_n \xrightarrow{n \to \infty} x_0$, we have $\lim\limits_{n \to \infty} f(a_n) = f(x_0)$

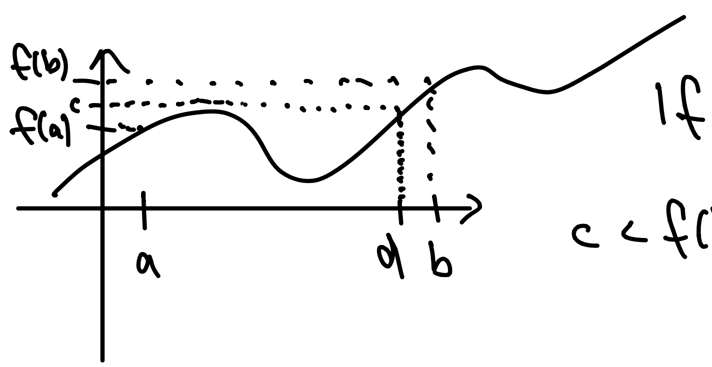
Ex.: $f(x) = \begin{cases} 1, & x > 0 \\ -1, & x \leq 0 \end{cases}$, $x_0 = 0$

take seq. $\left(-\frac{1}{n}\right)_{n\in\mathbb{N}} \xrightarrow{n\to\infty} 0$, $\quad f\left(-\frac{1}{n}\right) = -1 = f(x_0)$

take seq. $\left(\frac{1}{n}\right)_{n\in\mathbb{N}} \xrightarrow{n\to\infty} 0$, $\quad f\left(\frac{1}{n}\right) = 1 \neq f(x_0) = -1$

Consequences (without proof):

- If $f, g: A \to \mathbb{R}$ are cont. at x_0, then so are $f+g$, λf $(\lambda \in \mathbb{R})$, $f \cdot g$, $\frac{f}{g}$ if $g(x_0) \neq 0$
- The composition of $f: A \to B$ and $g: C \to D$ with $D \subset A$ is def. as $f \circ g$, with $(f \circ g)(x) = f(g(x))$.
  If f and g are cont., then so is $f \circ g$.
- Intermediate Value Theorem:



If $f: [a,b] \to \mathbb{R}$ is cont. and $c > f(a)$, $c < f(b)$, then $\exists\, d \in [a,b]$ s.t. $f(d) = c$.

- Maximum Theorem: next time