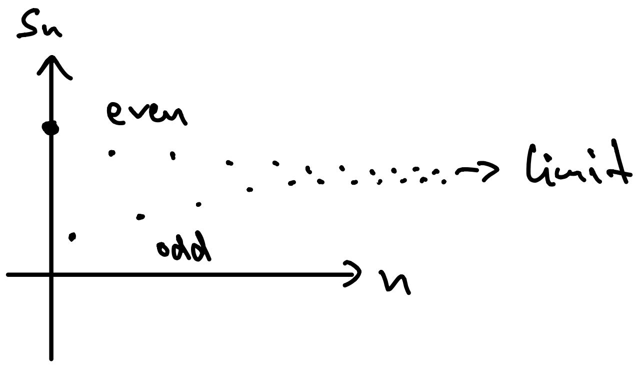
Session 6
Sep. 19, 2018

Leibniz: consider $\sum_{k=0}^{\infty} (-1)^k a_k$ with $a_k > 0 \;\forall\, k \geq 0$.

If $a_{k+1} \leq a_k$ and $\lim_{k\to\infty} a_k = 0$, then the series converges.

Proof:

idea:



odd partial sums: $S_{2n+1} = S_{2n-1} + \underbrace{a_{2n} - a_{2n+1}}_{\geq 0} \geq S_{2n-1}$

even partial sums: $S_{2n} = S_{2n-2} \underbrace{- a_{2n-1} + a_{2n}}_{\leq 0} \leq S_{2n-2}$

also $S_{2n+1} = S_{2n} - a_{2n+1} < S_{2n}$

$\Rightarrow \; s_1 \leq s_3 \leq s_5 \leq \ldots \leq s_4 \leq s_2 \leq s_0$

n odd: $S_n \leq S_{n+j} \leq S_{n+1} \;\; \forall\, j \geq 0$

n even: $S_{n+1} \leq S_{n+j} \leq S_n \;\; \forall\, j \geq 0$

n even: $S_{n+j} - S_n \leq 0$

carefully:

- n odd: $S_n \leq S_{n+j} \leq S_{n+1}$

$\Rightarrow S_{n+j} - S_n \geq 0$

$\Rightarrow |S_{n+j} - S_n| = S_{n+j} - S_n$

$\leq S_{n+1} - S_n \;\; (\geq 0)$

$= |S_{n+1} - S_n|$

- n even: similar

since $|S_{n+j}-S_n| \leq |S_{n+1}-S_n| = |a_{n+1}| \xrightarrow{n\to\infty} 0$, S_n is a Cauchy sequence, so it converges. $\square$

Ex.: $\sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ converges due to Leibniz criterion

now: magic trick

$$\underbrace{\sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1}}_{=:c} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots$$

$$= \underbrace{1-\frac{1}{2}}_{\frac{1}{2}}-\frac{1}{4}+\underbrace{\frac{1}{3}-\frac{1}{6}}_{\frac{1}{6}}-\frac{1}{8}+\underbrace{\frac{1}{5}-\frac{1}{10}}_{\frac{1}{10}}-\frac{1}{12}+\underbrace{\frac{1}{7}-\frac{1}{14}}_{\frac{1}{14}}-\ldots$$

$$= \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\frac{1}{14}-\ldots$$

$$= \frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\ldots\right)$$

$$= \frac{1}{2}\underbrace{\sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1}}_{=c} \qquad \Rightarrow c = \frac{1}{2}c$$

$$\Rightarrow c = 0$$

Later: $c = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1}$ is not zero!

Lesson: Limit can depend on the order of summation!

actually, by rearranging the terms above, one can let the series converge to any real number (Riemann, 1854)

Def.: $\sum_{k=0}^{\infty} a_k$ is absolutely convergent if $\sum_{k=0}^{\infty} |a_k|$ converges.

Thm.: If a series is absolutely convergent, then all rearrangements have the same limit.

Proof: omitted here

we will see that $\sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1}$ is not absolutely convergent

$$\left(\sum_{k=0}^{\infty} \left| (-1)^k \frac{1}{k+1} \right| = \sum_{k=0}^{\infty} \frac{1}{k+1} \text{ is divergent} \right)$$

Def.: conditionally convergent $\Longleftrightarrow$ convergent, but not absolutely

Convergence criteria for $\sum_{k=0}^{\infty} a_k$:

- necessary condition: $\lim_{k\to\infty} a_k = 0$

- Leibniz

- comparison test: $0 \le a_k \le b_k \quad \forall k \geq 0 \ (\forall k \geq m$ for some $m \in \mathbb{N})$

  Then: $\sum_{k=0}^{\infty} b_k$ conv. $\Rightarrow \sum_{k=0}^{\infty} a_k$ conv.

  $\sum_{k=0}^{\infty} a_k$ div. $\Rightarrow \sum_{k=0}^{\infty} b_k$ div.

Ex.: $\sum_{k=0}^{\infty} \frac{1}{k+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots$

let $\sum_{k=0}^{\infty} a_k = 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{=\frac{1}{2}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{=\frac{1}{2}} + \ldots \rightarrow$ diverges

$\Rightarrow$ by comparison, also $\sum_{k=0}^{\infty} \frac{1}{k+1}$ diverges

- ratio test: – If $\limsup_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1$ then $\sum_{k=0}^{\infty} a_k$ converges absolutely
  – If $\liminf_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| > 1$ then series diverges
  – If $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = 1$ then test is inconclusive

Ex.: $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ for any $x \in \mathbb{R}$

here: $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{x^{k+1}/(k+1)!}{x^k/k!}\right| = \frac{|x|}{k+1} \xrightarrow{k\to\infty} 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = 0$. Remember: If lim exists, it's equal to limsup = liminf.

$\Rightarrow$ converges absolutely

Why does this work?

say $a_k \geq 0 \ \forall k$, look at $\limsup_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$

there is some (very large) N s.t. $\frac{a_{m+1}}{a_m} \leq c < 1 \quad \forall m \geq N$
(otherwise the limsup could not be <1)

$$\Rightarrow \sum_{k=1}^{\infty} a_k = \underbrace{\sum_{k=1}^{N-1} a_k}_{\text{some number}} + \underbrace{a_N + a_{N+1} + a_{N+2} + \dots}_{\underbrace{a_N + c a_N + c^2 a_N + \dots}}$$

$$= a_N \sum_{j=0}^{\infty} c^j = a_N \frac{1}{1-c} \quad \text{, since } c < 1 \text{ (geom. series)}$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k \text{ converges}$$

- Root test: – If $\limsup_{k\to\infty} \sqrt[k]{|a_k|} < 1$, then abs. conv.
  – If $\limsup_{k\to\infty} \sqrt[k]{|a_k|} > 1$, then div.
  – If $\limsup_{k\to\infty} \sqrt[k]{|a_k|} = 1$, then inconclusive

Ex.: $\sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^k$, then $\sqrt[k]{\left|\frac{1}{k}\right|^k} = \left|\frac{1}{k}\right| = \frac{1}{k} \xrightarrow{k\to\infty} 0$

so $\sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^k$ conv. abs.