Session 7
Sep. 24, 2018

How do we multiply

$$\left(\sum_{k=0}^{\infty} a_k\right)\cdot\left(\sum_{k=0}^{\infty} b_k\right) = (a_0+a_1+a_2+a_3+\dots)\cdot(b_0+b_1+b_2+b_3+\dots) \; ?$$

$\longrightarrow$ many ways to multiply this out...

One possibility is the so-called Cauchy product:

$$\left(\sum_{k=0}^{\infty} a_k\right)\cdot\left(\sum_{k=0}^{\infty} b_k\right) := a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \dots$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} a_{n-j} b_j$$

<u>Thm.:</u> If $\sum_{k=0}^{\infty} a_k = a$ and $\sum_{k=0}^{\infty} b_k = b$ converge absolutely, then

$$a\cdot b = \left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right) = \underbrace{\sum_{n=0}^{\infty} \sum_{j=0}^{n} a_{n-j} b_j}_{\text{i.e., this series converges}}$$

1.6 Power Series and Exponential Function

$\sum_{k=0}^{\infty} a_k x^k$ is called a power series

For which x does it converge?

root test: convergence if $\limsup\limits_{k\to\infty} \sqrt[k]{\underbrace{|a_k x^k|}_{|a_k|\cdot|x|^k}} = \limsup\limits_{k\to\infty} \sqrt[k]{|a_k|}\,|x| < 1$

$\Rightarrow$ absolute convergence if $|x| < \dfrac{1}{\limsup\limits_{k\to\infty} \sqrt[k]{|a_k|}} =: \rho$

- divergence if $|x| > \rho$
- inconclusive if $|x| = \rho$ (might converge, might diverge, might be different for $x=\rho$ and $x=-\rho$)

ρ is called the $\underbrace{\text{radius}}$ of convergence

will make more sense for complex power series $\to$ later

note:

- ρ can be 0, >0 or ∞.
- we have $\rho = \sup\{|x| : \sum\limits_{k=0}^{\infty} a_k x^k \text{ converges}\}$
  - $\hookrightarrow$ $\sum\limits_{k=0}^{\infty} a_k x^k$ converges for $-\rho < x < \rho$
- alternatively we could use the ratio test:
  - $\hookrightarrow$ convergence if $\lim\limits_{k\to\infty} \left|\dfrac{a_{k+1}x^{k+1}}{a_k x^k}\right| = \lim\limits_{k\to\infty}\left|\dfrac{a_{k+1}}{a_k}\right| |x| < 1$, i.e.,

$|x| < \lim\limits_{k\to\infty}\left|\dfrac{a_k}{a_{k+1}}\right| \Rightarrow \rho = \lim\limits_{k\to\infty}\left|\dfrac{a_k}{a_{k+1}}\right|$ if this limit exists

(e.g., for $\frac{1}{2}, 0, (\frac{1}{2})^2, 0, (\frac{1}{2})^3, 0$ (every other element $=0$), limit doesn't exist)

Ex.: $P(x) = \sum\limits_{k=0}^{\infty} (2x)^k = 1 + 2x + 4x^2 + 8x^3 + \ldots$ $\quad (a_k = 2^k)$

$\rho = \lim\limits_{k\to\infty}\left|\dfrac{a_k}{a_{k+1}}\right| = \lim\limits_{k\to\infty}\left|\dfrac{2^k}{2^{k+1}}\right| = \frac{1}{2} \Rightarrow P(x)$ conv. for all $-\frac{1}{2} < x < \frac{1}{2}$

endpoints: $\cdot\, P(\frac{1}{2}) = \sum_{k=0}^{\infty} (2\cdot\frac{1}{2})^k = \sum_{k=0}^{\infty} 1$ diverges $\longrightarrow \infty$

$\cdot\, P(-\frac{1}{2}) = \sum_{k=0}^{\infty} (2\cdot(-\frac{1}{2}))^k = \sum_{k=0}^{\infty} (-1)^k$ diverges (oscillates)

note: $\cdot$ If conv. regions of power series $P(x)$ and $Q(x)$ overlap, also $P(x)+Q(x)$ and $P(x)\cdot Q(x)$ conv. (in overlapping region)

$\cdot$ If $P(x), Q(x)$ converge $\forall x$ $(\rho=\infty)$, then also $P(Q(x))$ conv. $\forall x$

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now: want to consider $(1+\frac{x}{n})^n$ (e.g., interest compounding)

what is $\lim_{n\to\infty} (1+\frac{x}{n})^n$ if it exists, depending on x?

binomial thm.:

$$\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{n}\right)^k \underbrace{1^{n-k}}_{=1} = \sum_{k=0}^{n} \frac{\binom{n}{k}}{n^k} x^k$$

now $$\frac{\binom{n}{k}}{n^k} = \frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{k!\,n^k} = \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdot\ldots\cdot\left(1-\frac{(k-1)}{n}\right)$$

$$\Rightarrow \left(1+\frac{x}{n}\right)^n = \sum_{k=1}^{n} \frac{x^k}{k!}\left(1-\frac{1}{n}\right)\cdot\ldots\cdot\left(1-\frac{(k-1)}{n}\right) + 1$$

$$= 1 + x + \frac{x^2}{2!}\left(1-\frac{1}{n}\right) + \frac{x^3}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots$$

guess: $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{\infty}\frac{x^k}{k!}$

need to be careful with $\lim_{n\to\infty}$ (e.g., $\lim_{n\to\infty}\underbrace{\sum_{k=1}^{n}\frac{1}{n}}_{=1} \neq \sum_{k=1}^{\infty} 0 = 0$)

but here, without proof, the guess is correct

$\Rightarrow \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{\infty}\frac{x^k}{k!}$ which converges absolutely $\forall x\in\mathbb{R}$ (ratio test).

now: $\sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{j=0}^{\infty}\frac{y^k}{k!} = \sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\frac{x^{n-\ell}}{(n-\ell)!}\frac{y^\ell}{\ell!}$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\underbrace{\sum_{\ell=0}^{n}\binom{n}{\ell}x^{n-\ell}y^{\ell}}_{=(x+y)^n}$$

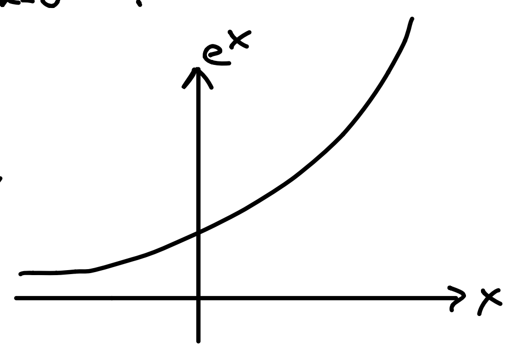
$$= \sum_{n=0}^{\infty}\frac{(x+y)^n}{n!}$$

so $f(x) = \sum_{k=0}^{\infty}\frac{x^k}{k!}$ fulfills $f(x)f(y) = f(x+y)$

$\Rightarrow f$ is exponentiation, i.e., $f(x) = e^x$ for some $e\in\mathbb{R}$.

To what base, i.e., what is e? $e = e^1 = f(1) = \sum_{k=0}^{\infty}\frac{1}{k!}$

e is called Euler's number, and $e \approx 2.718\ldots$



$\Rightarrow f(x) = e^x$ with property $e^x e^y = e^{x+y}$

$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$ for any $x\in\mathbb{R}$!