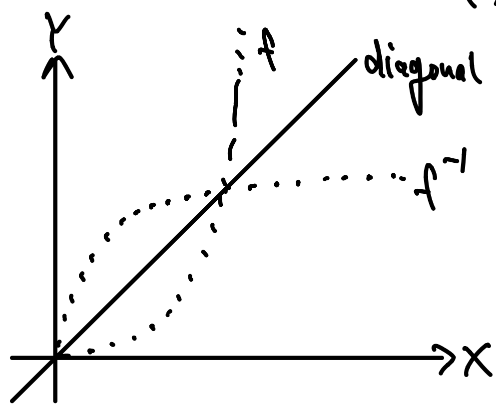
Session 8
Sep. 26, 2018

recall: bijective fct.s $f: A \to B$ have inverses

Inverse means: If $f(x)=y$ then $f^{-1}(y) = f^{-1}(f(x)) = x$.

$\hookrightarrow f^{-1}f = \mathrm{id} = \mathbb{1}$ (identity)



Ex.: $f: [0,\infty) \to [0,\infty), x \mapsto x^2$

$\hookrightarrow x^2 = y \Rightarrow x = \sqrt{y}$

$\Rightarrow f^{-1}: [0,\infty) \to [0,\infty), x \mapsto \sqrt{x}$

graph of f^{-1} = reflection of f across line $x=y$

$\exp: \mathbb{R} \to (0,\infty), x \mapsto e^x$ is actually bijective

inverse $\ln: (0,\infty) \to \mathbb{R}, x \mapsto \ln x$ is called natural logarithm

note: we can thus define a^x for any $a>0$ and any $x \in \mathbb{R}$:

$$a^x = \exp(\ln(a^x)) = \exp(x \ln a) = e^{x \ln a}$$

more generally: the inverse of a^x is called $\underbrace{\log_a x}_{\text{logarithm of } x \text{ to basis } a.}$, for any $a>0$

$\hookrightarrow$ see HW

note: $x^r = e^{\ln x^r} = e^{r \ln x} \xrightarrow{x \to 0} 0$, $\xrightarrow{r \to 0} e^0 = 1$

side note: useful notation: Landau symbols

e.g., convenient notation: $e^x = 1 + x + \frac{x^2}{2} + \underbrace{\mathcal{O}(x^3)}$ for small x

Landau symbol (big O): terms of order x^3

in general: $f(x) = \mathcal{O}(g(x))$ as $x \to 0$ (or $x \to \infty$ or any other number depending on context)

$\Leftrightarrow |f(x)| \leq C|g(x)|$ as $x \to 0$

Ex.: • $p(x) = 2x^5 + 3x^2 - 1 = \mathcal{O}(x^5)$ for large x

• $(1-x)^{-1} = 1 + x + x^2 + x^3 + \mathcal{O}(x^4)$ for small x

Landau symbol (small o): $f(x) = o(g(x))$ if $\left|\frac{f(x)}{g(x)}\right| \xrightarrow{\text{as } x \to 0} 0$

Ex.: • $p(x) = 2x^5 + 3x^2 - 1 = o(x^6)$ large x $(x \to \infty)$

• $(1-x)^{-1} = 1 + x + x^2 + o(x^2)$ small x $(x \to 0)$

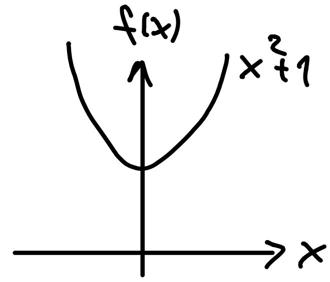
recall: geometric series

• $q(x) = 3x^4 + 2x^3 + 100x^2$

↳ then $p(x) \cdot q(x) = 6x^9 + \mathcal{O}(x^8)$ for large x

1.7 Complex Numbers

$f(x) = x^2 + 1 = 0$ has no real roots



$\Rightarrow$ introduce imaginary unit "$i := \sqrt{-1}$" or, rather, $i^2 = -1$

Ex.: Zeroes of $p(x) = x^2 + 2x + 10$

$\Rightarrow$ zeroes are $x_\pm = -1 \pm \sqrt{1-10} = -1 \pm \sqrt{-9} = -1 \pm 3\sqrt{-1}$

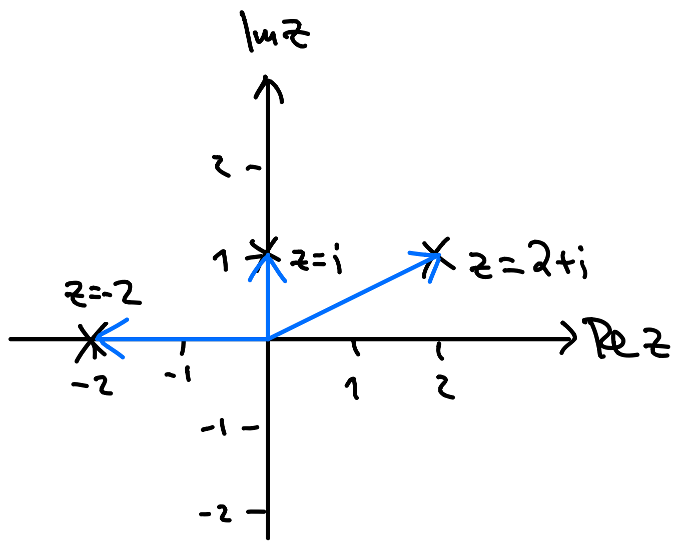
$= -1 \pm 3i$

Complex number $z = a + ib$, $a, b \in \mathbb{R}$

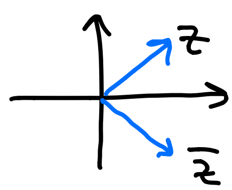
↳ a is called real part of z, $\operatorname{Re} z = a$

↳ b is called imaginary part of z, $\operatorname{Im} z = b$

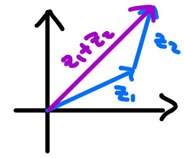
Complex plane (Argand diagram):



complex conjugate $\bar{z} = \overline{(a+ib)} := a - ib$



addition: $z_1+z_2=(a_1+ib_1)+(a_2+ib_2)=(a_1+a_2)+i(b_1+b_2)$



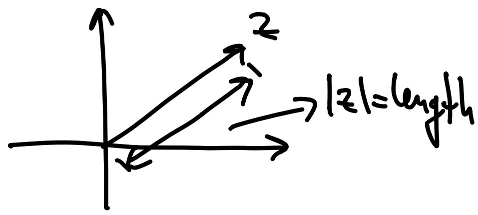
multiplication: $z_1\cdot z_2=(a_1+ib_1)\cdot(a_2+ib_2)=(a_1a_2-b_1b_2)+i(a_1b_2+a_2b_1)$

division: $\dfrac{z_1}{z_2}=\dfrac{a_1+ib_1}{a_2+ib_2}=\dfrac{z_1\overline{z_2}}{z_2\overline{z_2}}=\dfrac{(a_1+ib_1)(a_2-ib_2)}{(a_2+ib_2)(a_2-ib_2)}$

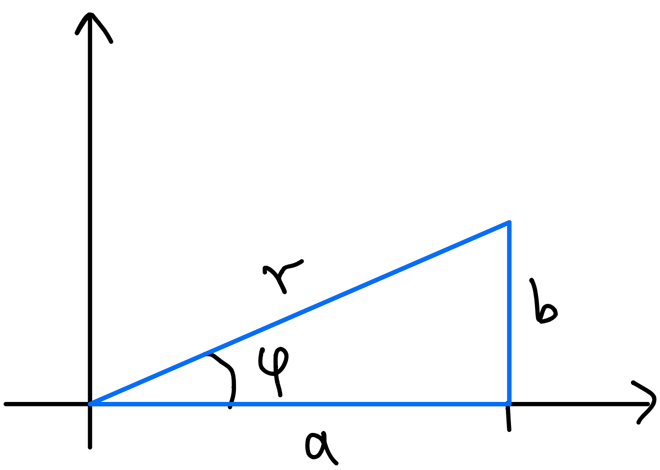
$$=\frac{a_1a_2+b_1b_2+i(-a_1b_2+a_2b_1)}{a_2^2+b_2^2}$$

$$=\left(\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}\right)+i\left(\frac{-a_1b_2+a_2b_1}{a_2^2+b_2^2}\right)$$

absolute value: $|z|:=\sqrt{z\bar{z}}=\sqrt{a^2+b^2}$



now: trigonometric functions



$\sin\varphi:=\dfrac{b}{r}$

$\cos\varphi:=\dfrac{a}{r}$

$\tan\varphi:=\dfrac{b}{a}$

Pythagoras: $r^2=a^2+b^2=r^2\cos^2\varphi+r^2\sin^2\varphi=r^2(\sin^2\varphi+\cos^2\varphi)$

$$\Rightarrow \sin^2\varphi+\cos^2\varphi=1$$

$\Rightarrow$ polar representation $z = r \cdot \cos\varphi + i\, r \cdot \sin\varphi$

$$= r\left(\cos\varphi + i \sin\varphi\right)$$

with $r = |z|$

and $\varphi = \begin{cases} \tan^{-1} \frac{\operatorname{Im} z}{\operatorname{Re} z} & \text{for } \operatorname{Re} z > 0 \\ \pi + \tan^{-1} \frac{\operatorname{Im} z}{\operatorname{Re} z} & \text{for } \operatorname{Re} z < 0 \end{cases}$ (φ is called "argument of z")

consider $e^{ix} = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$$= \underbrace{\left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}\right)}_{\text{even powers of } x} + i \underbrace{\left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}\right)}_{\text{odd powers of } x}$$

and $|e^{ix}| = \sqrt{e^{ix} e^{-ix}} = 1$

$\Rightarrow$ Guess (that we will deduce next time): $e^{ix} = \cos x + i \sin x$

Session 9
Oct. 1, 2018

we need to know series expansions of $\cos x$ and $\sin x$

easier later (with Taylor series), but here sketch of heuristic proof

from geometry, deduce
$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

Setting $y = nx$ gives us a recursion relation:
$$\sin((n+1)x) = \sin x \cos nx + \cos x \sin nx$$
$$\cos((n+1)x) = \cos x \cos nx - \sin x \sin nx$$

solving this recursion leads to de Moivre formulas:
$$\sin nx = \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} (-1)^{\frac{k-1}{2}} \binom{n}{k} \sin^k x \cos^{n-k} x$$

$$\cos nx = \sum_{\substack{k=0 \\ k \text{ even}}}^{n} (-1)^{\frac{k}{2}} \binom{n}{k} \sin^k x \cos^{n-k} x$$

Set $y = nx$, then
$$\sin y = \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} (-1)^{\frac{k-1}{2}} \underbrace{\binom{n}{k} \left(\sin \frac{y}{n}\right)^k \left(\cos \frac{y}{n}\right)^{n-k}}$$
$$\binom{n}{k}\left(\frac{y}{n}\right)^k = \frac{n!\, y^k}{(n-k)!\, k!\, n^k} = \frac{1}{k!} \frac{n(n-1)\cdot \ldots \cdot (n-k+1)}{n \cdot n \cdot \ldots \cdot n} y^k$$

use $\sin x \approx x$ and $\cos x \approx 1$ for small x (from geometry)

$$\xrightarrow{n \to \infty} \sum_{\substack{k=0 \\ k \text{ odd}}}^{\infty} (-1)^{\frac{k-1}{2}} \frac{y^k}{k!}$$, similar for $\cos y$