Session 9
Oct. 1, 2018

we need to know series expansions of $\cos x$ and $\sin x$

easier later (with Taylor series), but here sketch of heuristic proof

from geometry, deduce $\sin(x+y) = \sin x \cos y + \cos x \sin y$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

setting $y = nx$ gives us a recursion relation:

$$\sin((n+1)x) = \sin x \cos nx + \cos x \sin nx$$

$$\cos((n+1)x) = \cos x \cos nx - \sin x \sin nx$$

solving this recursion leads to de Moivre formulas:

$$\sin nx = \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} (-1)^{\frac{k-1}{2}} \binom{n}{k} \sin^k x \cos^{n-k} x$$

$$\cos nx = \sum_{\substack{k=0 \\ k \text{ even}}}^{n} (-1)^{\frac{k}{2}} \binom{n}{k} \sin^k x \cos^{n-k} x$$

set $y = nx$, then

$$\sin y = \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} (-1)^{\frac{k-1}{2}} \underbrace{\binom{n}{k} \left(\sin \frac{y}{n}\right)^k \left(\cos \frac{y}{n}\right)^{n-k}}$$

$$\binom{n}{k}\left(\frac{y}{n}\right)^k = \frac{n!\, y^k}{(n-k)!\, k!\, n^k} = \frac{1}{k!} \frac{n(n-1)\cdot\ldots\cdot(n-k+1)}{n\cdot n\cdot\ldots\cdot n} y^k$$

use $\sin x \approx x$ and $\cos x \approx 1$ for small x (from geometry)

$$\xrightarrow{n\to\infty} \sum_{\substack{k=0 \\ k \text{ odd}}}^{\infty} (-1)^{\frac{k-1}{2}} \frac{y^k}{k!}$$, similar for $\cos y$

this gives us Euler's formula $\boxed{e^{ix} = \cos x + i \sin x}$

$$\boxed{\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}}$$

$$(e^{i\pi} + 1 = 0)$$

$$\boxed{\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}}$$

$\Rightarrow$ any complex number can be written as $\boxed{z = r e^{i\varphi}}$ $(r \geq 0, 0 \leq \varphi < 2\pi)$

note: $\cdot\ (e^{i\varphi})^n = e^{in\varphi}$, so $(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i \sin n\varphi$

(de Moivre)

$\cdot$ say $n=2$: $(\cos\varphi + i\sin\varphi)^2 = \underline{\cos 2\varphi} + \underline{i \sin 2\varphi}$

$$\| $$

$$\underline{\cos^2\varphi} + \underline{2i\cos\varphi\sin\varphi} - \underline{\sin^2\varphi}$$

real and imaginary parts have to be the same

$$\Rightarrow \cos 2\varphi = \cos^2\varphi - \sin^2\varphi$$

$$\Rightarrow \sin 2\varphi = 2\cos\varphi\sin\varphi$$

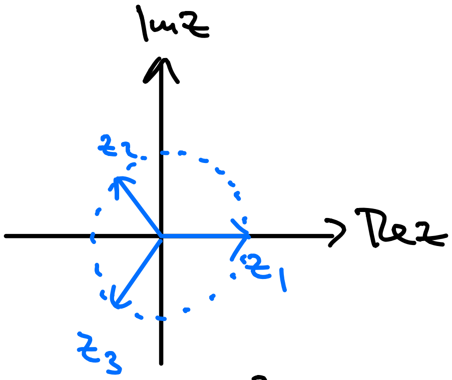
complex roots of $z^3 - 1 = 0$ $\left(\text{"}z = \sqrt[3]{1}\text{"}\right)$

write $1 = e^{2\pi i k}$ for $k = 0, 1, 2, 3, \ldots$

$$\Rightarrow 1^{\frac{1}{3}} = \left(e^{2\pi i k}\right)^{\frac{1}{3}} = e^{i\frac{2\pi k}{3}} = \begin{cases} 1 & , k=0 \\ e^{i\frac{2\pi}{3}} & , k=1 \\ e^{i\frac{4\pi}{3}} & , k=2 \end{cases}$$

for $k=3$ we're back at 1

$\Rightarrow z^3 = 1$ has three solutions $z_1 = 1$, $z_2 = e^{i\frac{2\pi}{3}}$, $z_3 = e^{i\frac{4\pi}{3}}$



$(z_3 = e^{i2\pi\frac{2}{3}})$

in gen. $e^{\frac{2\pi i k}{n}}$ for $k = 0, \dots, n-1$ are the n complex roots of $z^n = 1$.

similar: complex logarithm $\ln z$

write $z = re^{i\varphi + 2\pi i k}$, $k \in \mathbb{Z}$

$\ln z = \ln re^{i\varphi + 2\pi i k} = \ln r + i\varphi + 2\pi i k$, i.e., it has multiple values

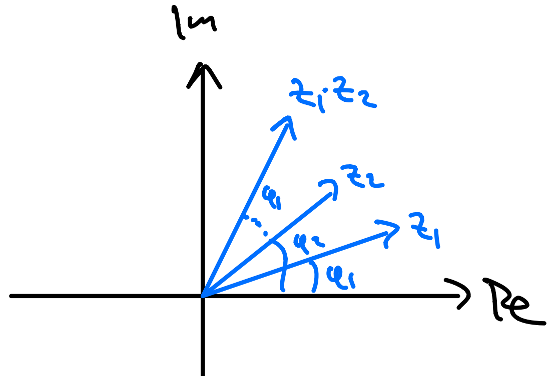
e.g., $\ln(i) = \ln\left(e^{i\frac{\pi}{2} + 2\pi i k}\right) = i\frac{\pi}{2} + 2\pi i k$, $k \in \mathbb{Z}$

the so-called principal value of $\ln z$ is def. by using $z = re^{i\varphi}$ with $-\pi < \varphi \leq \pi$

- multiplication: $z_1 = r_1 e^{i\varphi_1}$, $z_2 = r_2 e^{i\varphi_2}$

$\Rightarrow z_1 \cdot z_2 = r_1 r_2 e^{i(\varphi_1 + \varphi_2)}$

(stretching and rotating)



(note: $z^3 = 2 \Rightarrow z_1 = \sqrt[3]{2}$, $z_2 = \sqrt[3]{2}\, e^{i\frac{2\pi}{3}}$, $z_3 = \sqrt[3]{2}\, e^{i\frac{4\pi}{3}}$)