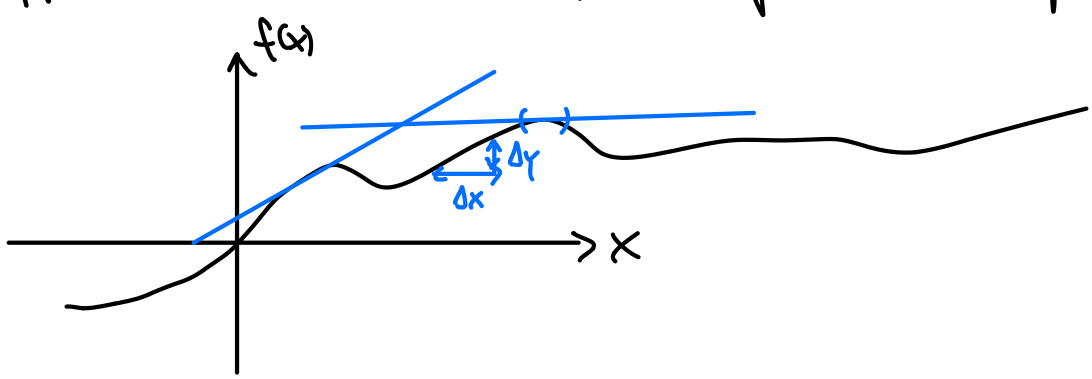
Session 10
Oct. 8, 2018

2. Derivatives and Applications

2.1 Basic Definition and Properties

Motivation:
find slope at each point of curve, or rather,

approximate curve by a linear fct. (tangent) at each point



"slope picture":

Ex.: • line $y = f(x) = ax + b$

$x \to x + \Delta x \, , \, y \to y + \Delta y$

$$\Rightarrow \underbrace{y + \Delta y}_{f(x) + \Delta y} = \underbrace{a(x + \Delta x) + b}_{f(x + \Delta x)} \Rightarrow \Delta y = a \Delta x \Rightarrow \text{slope } \frac{\Delta y}{\Delta x} = a$$

• parabola $y = f(x) = x^2$

$$\Rightarrow f(x) + \Delta y = f(x + \Delta x) = (x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$$

$$\Rightarrow \Delta y = 2x\Delta x + (\Delta x)^2 \Rightarrow \frac{\Delta y}{\Delta x} = 2x + \Delta x \xrightarrow{\Delta x \to 0} 2x$$

Def.: $f:[a,b]\to\mathbb{R}$ is differentiable at $x_0\in[a,b]$ if

$f(x)-f(x_0)=\Delta y=f(\Delta x+x_0)-f(x_0)$

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} \text{ exists.}$$

$x-x_0=\Delta x$

we call $f'(x_0)=\frac{df}{dx}(x_0)$ the derivative of f at x_0.

Remarks:

- note that we could also write $f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$
- we can give an equivalent definition inspired by "f is approximated by linear fct."

write definition above as $g_{x_0}=\frac{f(x)-f(x_0)}{x-x_0}-r(x)$ for some fct. $r(x)$ with $\lim_{x\to x_0} r(x)=0$.

If such g_{x_0} and $r(x)$ exist, then g_{x_0} is the derivative of f at x_0.

We can write this as $f(x)=\underbrace{f(x_0)+g_{x_0}\cdot(x-x_0)}_{\text{linear approximation}}+\underbrace{r(x)\cdot(x-x_0)}_{\text{rest that vanishes when } x\to x_0 \text{ faster than linear}}$

note that this def. can be nicely generalized to higher dimension

- we say f is differentiable if it is differentiable for all $x_0\in[a,b]$.

Ex.:

- $f(x)=x^2$

$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=\lim_{h\to 0}\frac{(x_0+h)^2-x_0^2}{h}$$

$$=\lim_{h\to 0}\frac{2x_0h+h^2}{h}=2x_0+\lim_{h\to 0}h=2x_0$$

- $f(x)=|x|$

$x_0>0$: $f'(x_0)=\lim_{h\to 0}\frac{|x_0+h|-|x_0|}{h}=\lim_{h\to 0}\frac{x_0+h-x_0}{h}=1$

$x_0 < 0$: $f'(x_0) = -1$

$x_0 = 0$: $f'(0) = \lim_{h \to 0} \frac{|h|}{h}$ doesn't exist $\Rightarrow$ f is not differentiable at 0

Differentiation Rules:

- f differentiable $\Longrightarrow$ f continuous
- $(f+g)' = f' + g'$
- product rule: $(fg)' = fg' + f'g$

Proof: $$(fg)'(x) = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} f(x+h) \frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} g(x) \frac{f(x+h) - f(x)}{h}$$

$$= f(x)g'(x) + g(x)f'(x)$$

Ex.: $(x^3)' = (x \cdot x^2)' = x \cdot 2x + 1 \cdot x^2 = 3x^2$

$\Rightarrow (x^n)' = nx^{n-1}$ for all $n \in \mathbb{N}$

- quotient rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ (when $g \neq 0$)

proof similar to above

Ex.: $(x^{-n})' = \left(\frac{1}{x^n}\right)' = \frac{-1 \cdot nx^{n-1}}{x^{2n}} = -nx^{-n-1}$

$\Rightarrow (x^m)' = mx^{m-1}$ for any $m \in \mathbb{Z}$

• chain rule: Let $k(x) = f(g(x))$ $(k = f \circ g)$. Then $k'(x) = f'(g(x)) \cdot g'(x)$

"$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$"

Proof: $$\frac{dk(x)}{dx} = \lim_{h\to 0} \frac{k(x+h)-k(x)}{h} = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h\to 0} \left(\frac{f(g(x+h)) - f(g(x))}{g(x+h)-g(x)}\right)\left(\frac{g(x+h)-g(x)}{h}\right)$$

$$= f'(g(x)) \cdot g'(x)$$

Ex.: $((x^3+4)^5)' = 5(x^3+4)^4 (x^3+4)' = 15x^2(x^3+4)^4$

• inverse fct.: f bijective, cont., and differentiable at x_0 and $f'(x_0) \neq 0$,

then $$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))} \qquad (f(x_0) = y_0)$$

Proof: $$\frac{df^{-1}(y_0)}{dy_0} = \lim_{y\to y_0} \frac{f^{-1}(y_0) - f^{-1}(y)}{y_0 - y}$$

$$= \lim_{f(x)\to f(x_0)} \frac{x_0 - x}{f(x_0) - f(x)}$$

$$= \lim_{x\to x_0} \frac{1}{\frac{f(x_0)-f(x)}{x_0 - x}} = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

Ex.: $y = f(x) = \sqrt{x} \Rightarrow x = y^2$

$$f'(x) = (\sqrt{x})' = \frac{1}{(y^2)'} = \frac{1}{2y} = \frac{1}{2\sqrt{x}}$$

$\Rightarrow$ in gen.: $(x^q)' = q x^{q-1}$ for all $q \in \mathbb{Q}$

more examples:

$$(e^x)' = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = e^x \underbrace{\lim_{h\to 0} \frac{e^h - 1}{h}}_{\lim_{h\to 0} \frac{1+h+\frac{1}{2}h^2+\mathcal{O}(h^3)-1}{h} = 1} = e^x$$

$$(\ln x)' = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$