Session 11
Oct. 10, 2018

for any $a \in \mathbb{R}$, $(x^a)' = \left(e^{a \ln x}\right)' = \underbrace{e^{a\ln x}}_{x^a} \underbrace{(a \ln x)'}_{\frac{a}{x}} = a x^{a-1}$

$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

$$= \sin(x) \underbrace{\lim_{h\to 0} \frac{\cos(h)-1}{h}}_{\lim_{h\to 0} \frac{1-\frac{h^2}{2}+O(h^3)-1}{h} = 0} + \cos(x) \underbrace{\lim_{h\to 0}\frac{\sin(h)}{h}}_{\lim_{h\to 0} \frac{h - \frac{h^3}{6} + O(h^4)}{h} = 1}$$

$$= \cos(x)$$

Derivative of Power Series:

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ have radius of convergence ρ

Then $\sum_{k=1}^{\infty} k a_k x^{k-1}$ has the same radius of convergence ρ

$$\left(\text{"root test"}: \limsup_{k\to\infty} \sqrt[k]{|k a_k|} = \limsup_{k\to\infty} \underbrace{\sqrt[k]{|k|}}_{=1 \text{ (see later)}} \sqrt[k]{|a_k|} = \rho \right)$$

$$\left(\text{"ratio test" (if it works)}: \lim_{k\to\infty} \left| \frac{k a_k}{(k+1)a_{k+1}} \right| = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| \right)$$

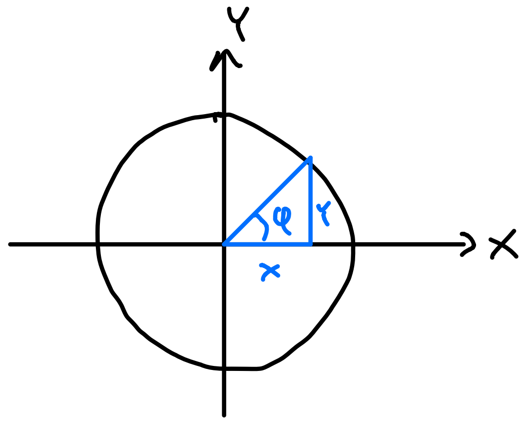
and indeed $f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$

Ex.: $(\sin x)' = \left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \right)' = \sum_{k=0}^{\infty} (-1)^k (2k+1) \frac{x^{2k}}{(2k+1)!}$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = \cos x$$

2.2 Implicit Differentiation and Parametric Representation

Ex.: circle



implicit equation $x^2+y^2=1$

parametrization: $x = \cos\varphi$ (φ parameter, $0 \le \varphi < 2\pi$)

$y = \sin\varphi$

task: find $\frac{dy}{dx}$

(1) solve explicitly: $y = \pm\sqrt{1-x^2}$

$$\frac{dy}{dx} = \pm\left(\frac{-2x}{2\sqrt{1-x^2}}\right) = \frac{-x}{\pm\sqrt{1-x^2}} = -\frac{x}{y}$$

but this strategy is often not possible or feasible, e.g., $x^3 - 3xy + y^3 = 2$

(2) implicit differentiation = take derivative on both sides of eq.

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1) = 0 \qquad (y = y(x))$$

$$\shortparallel$$

$$2x + 2y\frac{dy}{dx} \implies \frac{dy}{dx} = \frac{-x}{y}$$

Ex.: $\frac{d}{dx}\left(x^3-3xy+y^3\right)=0$

$$= 3x^2 - 3y - 3x\frac{dy}{dx} + 3y^2\frac{dy}{dx}$$

$$\left(\frac{d}{dx}(xy) = \underbrace{\frac{dx}{dx}}_{=1}y + x\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y-x^2}{y^2-x}$$

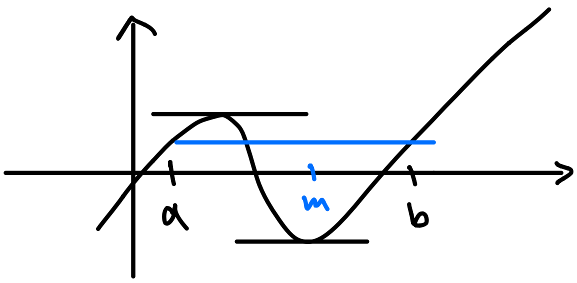
(3) via parametrization:

$$\frac{dy}{dx} = \underbrace{\frac{d\sin\varphi}{d\varphi}}_{\cos\varphi} \cdot \underbrace{\frac{d\varphi}{dx}}_{=\frac{1}{\frac{dx}{d\varphi}} = -\frac{1}{\sin\varphi}} = -\frac{\cos\varphi}{\sin\varphi} = -\frac{x}{y}$$

2.3 A Few Theorems about Derivatives

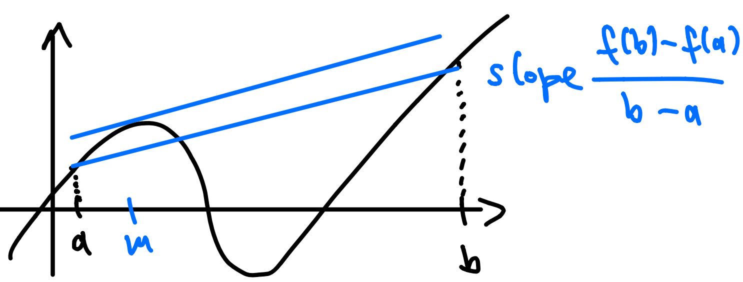
let $f: [a,b] \to \mathbb{R}$ continuous and differentiable on (a,b)

- Rolle: If $f(a)=f(b)$ then there is an $m \in (a,b)$ s.t. $f'(m)=0$.



Proof: maximum theorem + $f'(m)=0$ at maxima and minima.

- Lagrange: There is an $m \in (a,b)$ s.t. $f(b)-f(a) = f'(m)(b-a)$



Proof: define $h(x) = f(x) - \left(f(a) + (x-a)\frac{f(b)-f(a)}{b-a} \right)$

$\Rightarrow h(a) = 0 = h(b)$

Rolle $\Rightarrow \exists\, m$ s.t. $h'(m) = 0$, $h'(m) = f'(m) - \frac{f(b)-f(a)}{b-a}$.

so $f'(m) - \frac{f(b)-f(a)}{b-a} = 0$.

- Cauchy: take f, g both continuous on $[a,b]$ and differentiable on (a,b):

let $g'(x) \neq 0 \ \forall x \in (a,b)$. Then $g(a) \neq g(b)$ and $\exists\, m \in (a,b)$ s.t.

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(m)}{g'(m)}.$$

Proof: as above with $h(x) = f(x) - \left(f(a) + (g(x)-g(a))\frac{f(b)-f(a)}{g(b)-g(a)} \right)$

Important consequence:

<u>L'Hospital Thm.:</u> let $f, g : (a,b) \to \mathbb{R}$ be differentiable, $g'(x) \neq 0 \ \forall x \in (a,b)$

If $\lim_{x\to b^-} f(x) = 0$ and $\lim_{x\to b^-} g(x) = 0$ and $\lim_{x\to b^-} \frac{f'(x)}{g'(x)}$ exists, then

$$\boxed{\lim_{x\to b^-} \frac{f(x)}{g(x)} = \lim_{x\to b^-} \frac{f'(x)}{g'(x)}}.$$

note: same works for
- $f(x) \xrightarrow{x\to b^-} \infty$ and $g(x) \xrightarrow{x\to b^-} \infty$
- $b = \infty$
- $\lim_{x\to a^+}$