Session 13
Oct. 17, 2018

Ex.: $f(x) = \sin x$

$$\Rightarrow f'(x) = \cos x$$
$$f''(x) = -\sin x$$
$$f'''(x) = -\cos x$$
$$f^{(4)}(x) = \sin x$$
$$\vdots$$

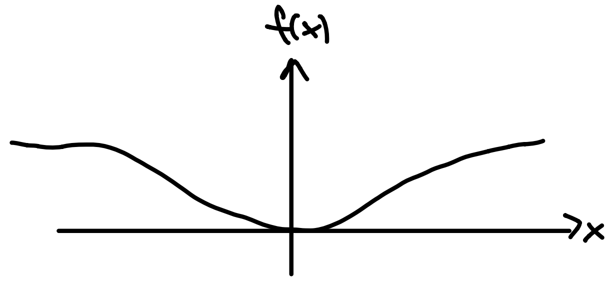
Taylor series around $a=0$, $\sin(0)=0$, $\cos(0)=1$

$$\Rightarrow f^{(k)}(0) = \begin{cases} 0 & \text{for } k \text{ even} \\ (-1)^{\frac{k-1}{2}} & \text{for } k \text{ odd} \end{cases}$$

remainder: $\left| \frac{x^n}{n!} f^{(n)}(m_x) \right| \le \frac{x^n}{n!} \xrightarrow{n\to\infty} 0$ for any x

$$\Rightarrow \sin x = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} (-1)^{\frac{k-1}{2}} \frac{x^k}{k!} \quad \text{(see before)}$$

Ex.: $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$



note: $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} e^{-\frac{1}{x^2}} = 0 = f(0)$, so f is continuous

$$f'(x) = \frac{2}{x^3} e^{-\frac{1}{x^2}}, \quad x \neq 0$$

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{e^{-\frac{1}{h^2}}}{h} = \lim_{y\to\infty} y \cdot e^{-y^2} = 0$$

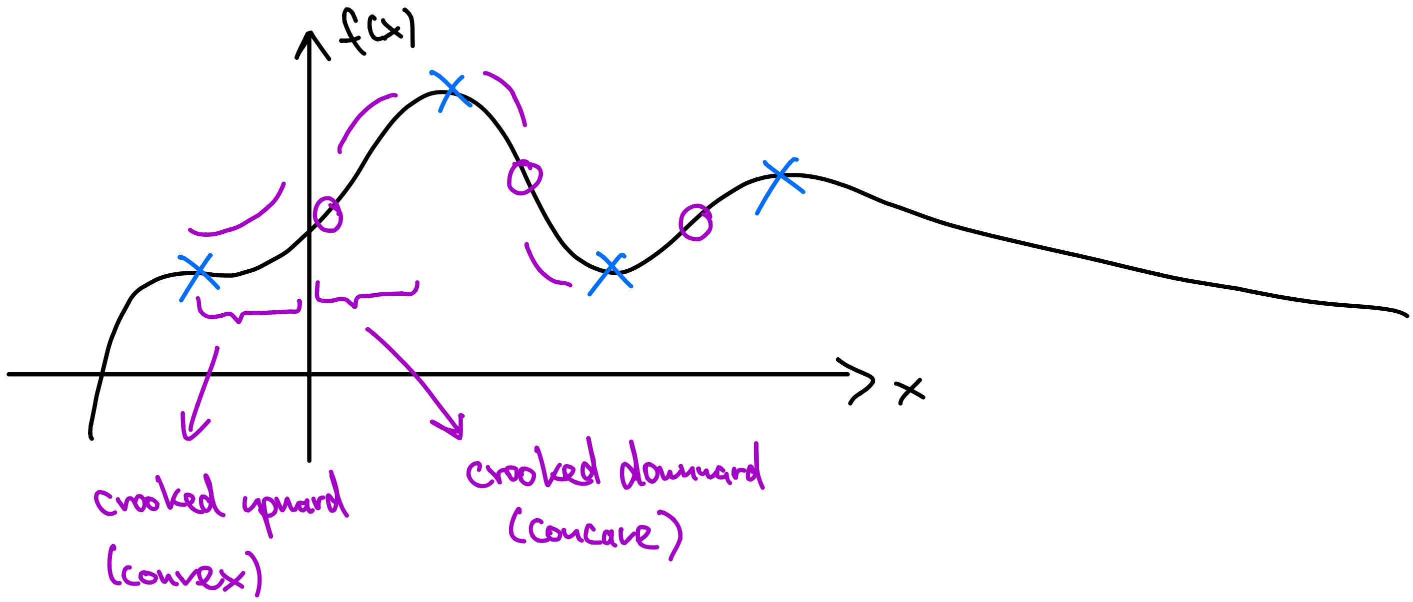
$\lim\limits_{x\to 0} f'(x) = \lim\limits_{x\to 0} \frac{2}{x^3} e^{-\frac{1}{x^2}} = 0$, so f' exists everywhere and is continuous next: $f''(x) = \left(\frac{2}{x^3}\right)' e^{-\frac{1}{x^2}} + \frac{2}{x^3}\left(e^{-\frac{1}{x^2}}\right)'$

$$= \frac{-6}{x^4} e^{-\frac{1}{x^2}} + \frac{2}{x^3} \cdot \frac{2}{x^3} e^{-\frac{1}{x^2}} = \left(\frac{-6}{x^4} + \frac{4}{x^6}\right) e^{-\frac{1}{x^2}}$$

then by similar argument: $f^{(k)}(0) = 0 \quad \forall k \geq 1$.

$\Rightarrow$ Taylor series $= 0 \neq f(x)$, since remainder nowhere small

2.5 Minimization and Maximization Problems



If $f'(x_0) = 0$ then x_0 is called stationary point

- If $f'(x_0) = 0$ and $f'(x_0)$ changes sign around $x_0 \Rightarrow$ max. or min.
  - ↳ from + to − $\Rightarrow$ max.
  - ↳ from − to + $\Rightarrow$ min.
- $f'(x_0)$ and $f''(x_0) > 0$ then x_0 is a local minimum
- $f'(x_0)$ and $f''(x_0) < 0$ then x_0 is a local maximum

• If $f''(x_0)=0$ and $f''(x_0)$ changes sign around x_0, then x_0 is called **point of inflection**

Ex.: $f(x) = x^3 - x^2 - 3x + 1$

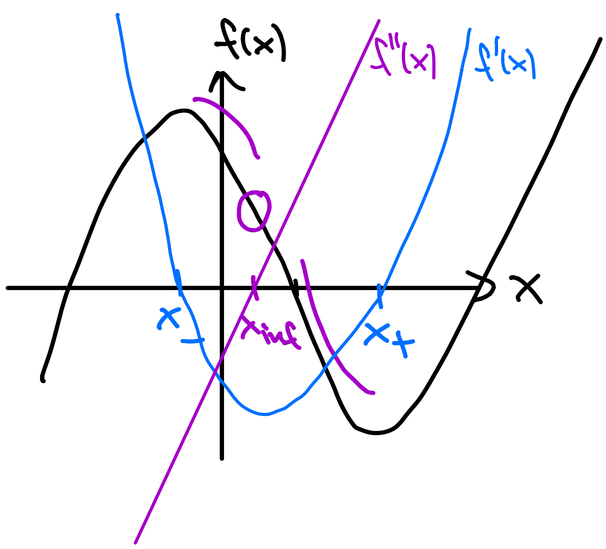
$f'(x) = 3x^2 - 2x - 3$

$f''(x) = 6x - 2$

$0 = f'(x) = 3x^2 - 2x - 3 = 3\left(x^2 - \frac{2}{3}x - 1\right) \Rightarrow x_\pm = \frac{1}{3} \pm \sqrt{\frac{1}{9} + 1}$

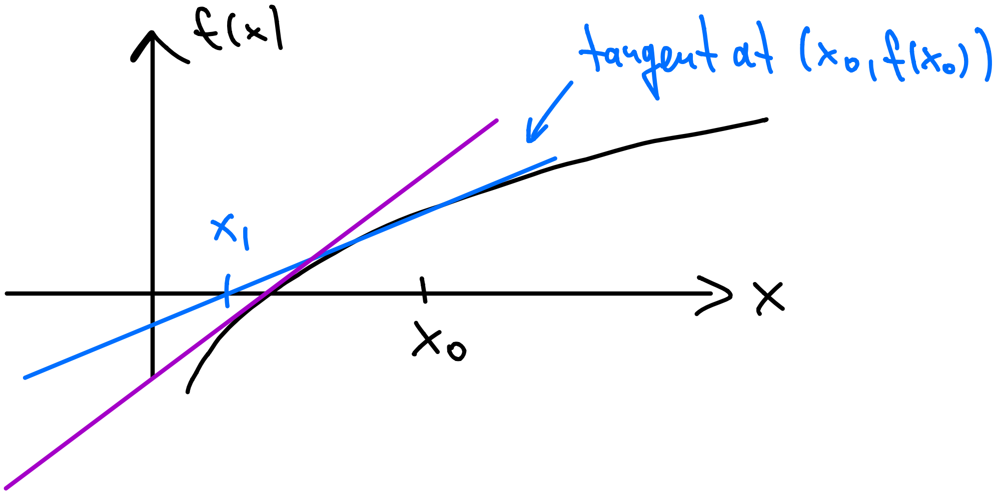
$= \frac{1}{3} \pm \frac{\sqrt{10}}{3}$

$0 = f''(x) = 6x - 2 \Rightarrow x_{inf} = \frac{1}{3}$



2.6 Newton's Method

finding roots by iteration (useful for numerical computation but also an important theoretical tool)



tangent through $(x_0, f(x_0))$ is $y(x) = f'(x_0)(x - x_0) + f(x_0)$

find x_1 by setting $y(x) = 0$

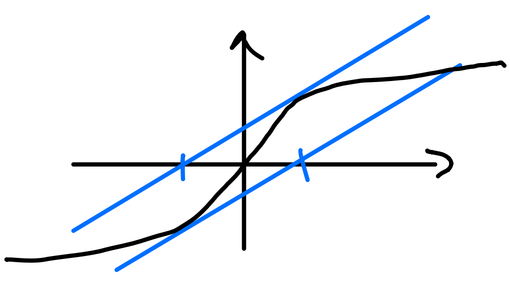
$\Rightarrow f'(x_0)(x_1 - x_0) + f(x_0) = 0$

$$\Rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

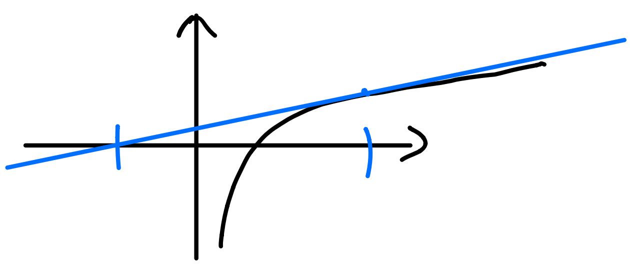
repeating this leads to iteration $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

↳ this is called Newton or Newton-Raphson method

the method can fail, i.e., iteration need not converge to a root, e.g.,



$x_{n+2} = x_n$



$f(x_{n+1})$ not defined

or $f'(x_n) = 0$ for some n

Ex.: $x^2 - a = 0$

$$\Rightarrow x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

e.g., $a=2$, $x_0 = 1 \Rightarrow x_1 = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2}$

$\Rightarrow x_2 = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right) = \frac{17}{12} \approx 1.417\ldots$ already close to $\sqrt{2} = 1.414\ldots$

next time: how fast is the convergence?