Session 14
Oct. 29, 2018

If an iteration scheme converges, how fast?

gen. iteration scheme: $x_{n+1} = F(x_n)$

convergence means we are looking for a fixed point, i.e., z s.t. $z = F(z)$
$(z = \lim_{n\to\infty} x_n)$

Ex.: Newton: $F(x) = x - \frac{f(x)}{f'(x)}$

zero at $z \Rightarrow f(z) = 0 \Rightarrow F(z) = z$

consider $\varepsilon_n = x_n - z$

$$\Rightarrow x_{n+1} = z + \varepsilon_{n+1} = F(x_n) = F(z + \varepsilon_n)$$

Taylor expansion: $F(z+\varepsilon_n) = \underbrace{F(z)}_{=z} + \varepsilon_n F'(z) + \frac{\varepsilon_n^2}{2} F''(z) + O(\varepsilon_n^3)$
$= z + \varepsilon_{n+1}$

(remember: Taylor: $f(x) = f(a) + (x-a)f'(a) + \dots$
or $f(x+a) = f(a) + x f'(a) + \dots$)

$$\Rightarrow \varepsilon_{n+1} = \varepsilon_n F'(z) + \frac{\varepsilon_n^2}{2} F''(z) + O(\varepsilon_n^3)$$

for decreasing error: want $\left|\frac{\varepsilon_{n+1}}{\varepsilon_n}\right| \approx |F'(z)| < 1$

suppose $F'(z) = 0 \Rightarrow \varepsilon_{n+1} = \frac{\varepsilon_n^2}{2} F''(z) + O(\varepsilon_n^3)$

then $\frac{\varepsilon_{n+1}}{\varepsilon_n} = O(\varepsilon_n)$ or $\varepsilon_{n+1} = O(\varepsilon_n^2)$, so convergence is much faster (quadratic conv.)

in gen.: if $\mathcal{F}^{(k)}(z) = 0 \ \forall \ k=1,\dots,N-1$ then $\varepsilon_{n+1} = \mathcal{O}(\varepsilon_n^N)$

this is called N-th order convergence

Newton: $\mathcal{F}(x) = x - \dfrac{f(x)}{f'(x)}$

$$\Rightarrow \mathcal{F}'(x) = 1 - \left[\frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2}\right] = \frac{f(x)f''(x)}{f'(x)^2}$$

$\Rightarrow \mathcal{F}'(z) = \dfrac{f(z)f''(z)}{f'(z)^2} = 0$ since $f(z)=0$, as long as $f'(z) \neq 0$.

$$\text{also: } \mathcal{F}''(x) = \frac{\left(f'(x)f''(x) + f(x)f'''(x)\right)f'(x)^2 - f(x)f''(x)2f'(x)f''(x)}{f'(x)^4}$$

$$= \frac{f''(x)f'(x)^2 + f(x)f'''(x)f'(x) - 2f(x)f''(x)^2}{f'(x)^3}$$

so $\mathcal{F}''(z) = \dfrac{f''(z)}{f'(z)} \neq 0$ if $f''(z) \neq 0$

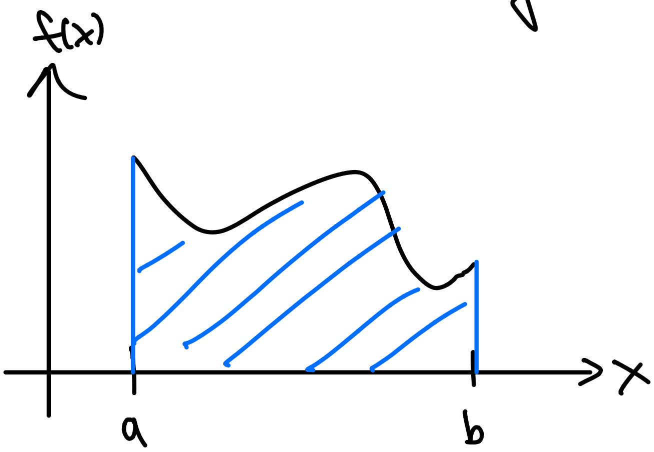
$\Rightarrow$ If Newton's method converges and $f'(z) \neq 0$, it does so quadratically

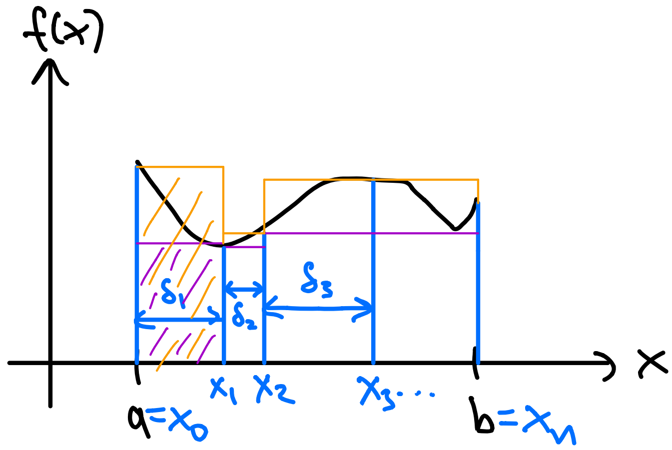
3. Integrals and Applications

3.1 Basic Definition and Properties

$f: [a,b] \to \mathbb{R}$ bounded

we want to define integral as the area between fct. and x-axis





division of $[a,b]$: $a=x_0<x_1<x_2<\dots<x_n=b$, $\delta_k = x_k - x_{k-1}$

lower Darboux sum: $S^{low} = \sum_{k=1}^{n} f_k^{low} \delta_k$ with $f_k^{low} = \inf_{x\in[x_{k-1},x_k]} f(x)$

upper Darboux sum: $S^{up} = \sum_{k=1}^{n} f_k^{up} \delta_k$ with $f_k^{up} = \sup_{x\in[x_{k-1},x_k]} f(x)$

note: • finer division $\Rightarrow$ S^{low} becomes bigger (or stays the same)

S^{up} becomes smaller (or stays the same)

• $S^{low} \leq S^{up}$

upper integral : $\overline{\int_a^b} f(x)dx := \inf_{\text{divisions}} S^{up}$

lower integral: $\underline{\int_a^b} f(x)dx := \sup_{\text{divisions}} S^{low}$

If upper and lower integral are equal, then f is Riemann integrable and the Riemann integral is $\int_a^b f(x)dx \ \left(= \overline{\int} = \underline{\int} \right)$.

Ex.: $f(x) = x$

choose $x_k = a + k\left(\frac{b-a}{n}\right)$, $k = 0, \dots, n$, $\delta_k = \frac{b-a}{n}$

$$f_k^{low} = \inf_{x \in [x_{k-1}, x_k]} x = a + (k-1)\left(\frac{b-a}{n}\right), \quad f_k^{up} = a + k\left(\frac{b-a}{n}\right)$$

$$S^{low} = \sum_{k=1}^{n} f_k^{low} \delta_k = \sum_{k=1}^{n} \left(a + (k-1)\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right), \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

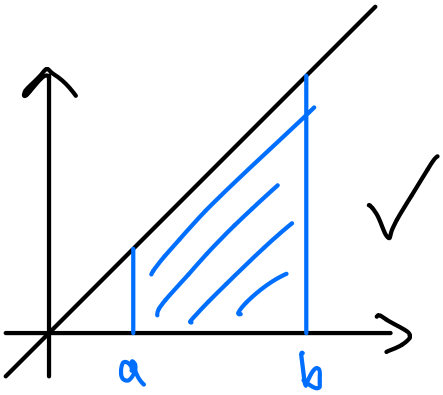
$$= \left[n \cdot a + \left(\frac{n(n+1)}{2} - n\right)\left(\frac{b-a}{n}\right)\right]\left(\frac{b-a}{n}\right)$$

$$= ab - a^2 + \frac{1}{2}\left(1 - \frac{1}{n}\right)\underbrace{(b-a)^2}_{=b^2 - 2ab + a^2}$$

$$= \frac{b^2}{2} - \frac{a^2}{2} - \frac{(b-a)^2}{2n}$$

$$S^{up} = S^{low} + \frac{(b-a)^2}{n} = \frac{b^2}{2} - \frac{a^2}{2} + \frac{(b-a)^2}{2n}$$

$\lim\limits_{n\to\infty} S^{up} = \lim\limits_{n\to\infty} S^{low} \Rightarrow f$ is integrable and $\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}$.



<u>Ex.:</u> $f(x) = \begin{cases} 1 & \text{for } x \in \mathbb{Q} \text{ (x rational)} \\ 0 & \text{for } x \notin \mathbb{Q} \text{ (x irrational)} \end{cases}$ on $[0,1]$

in any interval $[x_{k-1}, x_k]$ there is always a rational and an irrational number

$\Rightarrow f_k^{up} = 1$, $f_k^{low} = 0$, $S^{up} = \sum_{k=1}^{n} f_k^{up} \delta_k = \sum_{k=1}^{n} \delta_k = 1$, $S^{low} = 0$

$\Rightarrow \overline{\int_0^1} f(x)\,dx = 1$, but $\underline{\int_0^1} f(x)\,dx = 0$, so f is not Riemann integrable