Session 15
Nov. 5, 2018

Some general theorems:

- f, g integrable, $\lambda \in \mathbb{R}$
  $\Rightarrow f+g, \lambda f, f \cdot g, \frac{f}{g}$ (if $|g(x)| \geq c > 0$) are integrable

- f continuous $\Rightarrow f$ integrable

- $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx \quad \forall a,b,c$

  $\hookrightarrow$ in particular: $\cdot \int_a^a f(x)dx = 0$

  $\cdot \int_a^b f(x)dx = -\int_b^a f(x)dx$

- mean-value thm.: f continuous, g integrable, $g > 0$. Then $\exists\, m \in [a,b]$

$$\int_a^b f(x)g(x)dx = f(m) \int_a^b g(x)dx.$$

in particular: if $g(x) = 1$ for all x ($g = 1$), then $\int_a^b f(x)dx = f(m)(b-a)$

Proof: let $f_{min} = \min_{x \in [a,b]} f(x)$, $f_{max} = \max_{x \in [a,b]} f(x)$

$\Rightarrow f_{min}\, g(x) \leq f(x)g(x) \leq f_{max}\, g(x)$

$\Rightarrow f_{min} \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq f_{max} \int_a^b g(x)dx$

$$\Rightarrow f_{min} \le \underbrace{\frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}}_{= f(m) \text{ for some } m \text{ because } f \text{ is continuous}} \le f_{max}$$

<u>Def.:</u> If $f(x)$ is derivative of some fct. $F(x)$ $(F'(x) = f(x))$, then $F(x)$ is called primitive or antiderivative of $f(x)$

<u>Fundamental Theorem of Differential Calculus:</u>

If $f: [a,b] \to \mathbb{R}$ is continuous. Then $\exists$ a primitive F of f, unique up to an additive constant and $\int_a^b f(x)dx = F(b) - F(a)$.

Note: $f(x)$ might not have a primitive, e.g., $f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$

for any $a > 0$

$$\Rightarrow \int_{-a}^{y} f(x)dx = \begin{cases} 0, & y < 0 \\ y, & y \ge 0 \end{cases}$$ not differentiable at zero.

<u>Proof:</u> let $F(x) = \int_a^x f(y)dy$

$$\Rightarrow F(x) - F(x_0) = \int_a^x f(y)dy - \int_a^{x_0} f(y)dy = \int_{x_0}^x f(y)dy$$

$\Rightarrow$ by mean-value thm.: $\exists\, u \in [x, x_0]$ s.t. $\int_{x_0}^x f(y)dy = f(u)(x - x_0)$

$$\Rightarrow \frac{F(x)-F(x_0)}{x-x_0} = f(u)$$

$$\Rightarrow \lim_{x\to x_0} \frac{F(x)-F(x_0)}{x-x_0} = f(x_0) \quad \left(u\in[x, x_0], \text{ so in the limit } u \to x_0\right)$$

now: use Lagrange thm. to show uniqueness up to an additive const.:

If $g'(u)=0 \ \forall u\in[a,b]$ then $g(b)-g(a) = g'(u)(b-a) = 0$

$\Rightarrow g(x) = C$ for all x

so if $g_1'(u) = g_2'(u) \ \forall u\in[a,b] \Rightarrow g_1(x) = g_2(x) + C$

$\Rightarrow F(x)+C$ for all constants C are exactly the primitives $\square$

3.1 Integration Techniques

Notation: $\int f(x)dx = F(x) + C$ meaning $\int_a^b f(x)dx = F(b)-F(a)$

- Integration by inspection (i.e., using $F'(x) = f(x)$)

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \forall n \neq -1$$

$$\int \frac{1}{x} dx = \ln x + C$$

$$\int e^x dx = e^x + C$$

$$\int \sin(x)\,dx = -\cos x + C$$

$$\int \cos(x)\,dx = \sin x + C$$

$$\int \tan(x)\,dx = -\ln(\cos(x)) + C \qquad \left(|x| < \frac{\pi}{2}\right)$$

$$\int \cos x\,(\sin x)^n\,dx = \frac{(\sin x)^{n+1}}{n+1} + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C \qquad (|x| < 1)$$

$$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos x + C \qquad (|x| < 1)$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + C \qquad (\text{when } f > 0)$$

- Substitution

consider $F(g(y))$ => chain rule: $\frac{d}{dy} F(g(y)) = F'(g(y)) \cdot g'(y)$

if $F' = f$ => $\int_a^b f(g(y))\,g'(y)\,dy = F(g(b)) - F(g(a))$

$$= \int_{g(a)}^{g(b)} f(x)\,dx$$

or: $\int_c^d f(x)\,dx \overset{x=g(y)}{=} \int_{g^{-1}(c)}^{g^{-1}(d)} f(g(y))\,g'(y)\,dy$

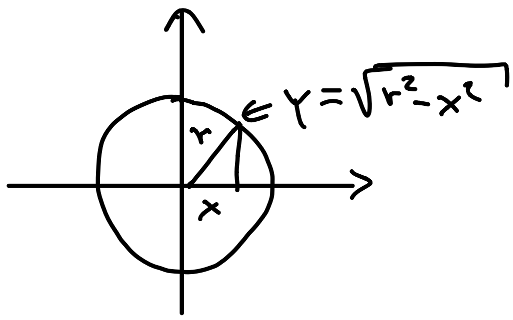
Ex.: • $\int e^{3x+1}\,dx$, substitute $3x+1=y$ or $x=\frac{1}{3}(y-1)=g(y)$

and $g'(y)=\frac{1}{3}$

$$\parallel$$

$$\int e^{y}\tfrac{1}{3}\,dy=\tfrac{1}{3}e^{y}+C=\tfrac{1}{3}e^{3x+1}+C$$

• area of a disc



$$\Rightarrow \text{area } A_r=4\int_0^r\sqrt{r^2-x^2}\,dx=4r\int_0^r\sqrt{1-\left(\tfrac{x}{r}\right)^2}\,dx$$

$$\tfrac{x}{r}=y \quad =4r^2\int_0^1\sqrt{1-y^2}\,dy=r^2\cdot A_1$$

$\Rightarrow A_r=r^2A_1$ (nice "scaling" argument)

$$\int\sqrt{1-y^2}\,dy \underset{\substack{y=\sin x\\ \frac{dy}{dx}=\cos x}}{=} \int\sqrt{1-\sin^2 x}\,\cos x\,dx=\int\cos^2 x\,dx$$

Recall: $(\cos x+i\sin x)^2=(e^{ix})^2=e^{2ix}=\cos 2x+i\sin 2x$

$$\Rightarrow \cos^2 x=\tfrac{1}{2}(1+\cos 2x)$$

$$\sin 2x=2\sin x\cos x$$

$$\Rightarrow \int\cos^2 x\,dx=\int\tfrac{1}{2}(1+\cos 2x)\,dx=\tfrac{1}{2}\left(x+\tfrac{1}{2}\sin 2x\right)$$

$$= \frac{1}{2}(x + \sin x \cos x)$$

$$= \frac{1}{2}\left(\arcsin y + y \cos(\arcsin y)\right)$$

$$= \frac{1}{2}\left(\arcsin y + y\sqrt{1-y^2}\right)$$

$\Rightarrow$ Area of unit disc $A_1 = 4\int_0^1 \sqrt{1-x^2}\,dx$

$$= 4\,\frac{1}{2}\left(\arcsin y + y\sqrt{1-y^2}\right)\Big|_0^1$$

$$= 2\left(\frac{\pi}{2}+0\right) - 2(0+0)$$

$$= \pi$$

$\Rightarrow A_r = \pi r^2$