Session 16
Nov. 7, 2018

- Integration by parts:

product rule: $(fg)' = f'g + fg'$

$$\Rightarrow \int (f(x)g(x))' dx = f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

$$\Rightarrow \int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$

Ex.: • $\int (\cos x) x dx = \int (\sin x)' x dx$

$$= (\sin x) x - \int (\sin x) \underbrace{(x)'}_{=1} dx$$

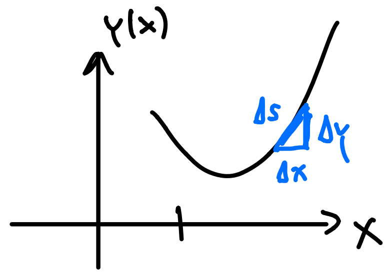
$$= x \sin x + \cos x$$

• $\int e^x x^2 dx = e^x x^2 - \int e^x 2x dx$

$$= e^x x^2 - 2\left[e^x x - \int e^x dx\right]$$

$$= e^x (x^2 - 2x + 2)$$

Ex.: curve length



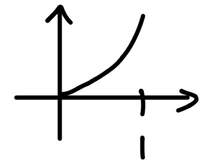
$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$

$$= (\Delta x)^2 \left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right)$$

infinitesimal segment of curve: " $ds = \sqrt{1 + (y'(x))^2}\, dx$ "

$$\Rightarrow \text{curve length } L = \int_a^b \sqrt{1 + (y'(x))^2}\, dx$$

e.g., parabola $y(x)=x^2$, curve length from 0 to 1



$$\Rightarrow y'(x)=2x \Rightarrow L=\int_0^1 \sqrt{1+4x^2}\,dx$$

$$2x=y \qquad =\frac{1}{2}\int_0^2 \sqrt{1+y^2}\,dy$$

computation of $L \to$ see HW

hints:

one way: $\sqrt{1+y^2} = \underbrace{1}_{\text{take primitive}} \cdot \sqrt{1+y^2} \Rightarrow$ express L as some fct. $+ \int \frac{1}{\sqrt{1+y^2}}\,dy$

then use substitution with $y=\sinh x$

useful since $\frac{d\sinh(x)}{dx}=\cosh x$ (check that) and $\cosh^2 x-\sinh^2 x=1$

- Recurrence Relations:

Ex.: $I_n=\int(\sin x)^n dx = \int \sin x(\sin x)^{n-1}dx$

$$=-\cos x(\sin x)^{n-1}-\int(-\cos x)(n-1)(\sin x)^{n-2}\cos x\,dx$$

$$=-\cos x(\sin x)^{n-1}+(n-1)\int\underbrace{(\cos x)^2}_{=1-(\sin x)^2}(\sin x)^{n-2}dx$$

$$=-\cos x(\sin x)^{n-1}+(n-1)\int(\sin x)^{n-2}dx-(n-1)\underbrace{\int(\sin x)^n dx}_{=I_n}$$

$$\Rightarrow I_n=\frac{-1}{n}\cos x(\sin x)^{n-1}+\left(\frac{n-1}{n}\right)\underbrace{\int(\sin x)^{n-2}dx}_{I_{n-2}}$$

repeat this until only $I_1 = \int \sin x \, dx = -\cos x$ or $I_0 = \int dx = x$ are left over

- Rational Functions:

Let $P(x), Q(x)$ be two polynomials with degrees $\deg P, \deg Q$

then $R(x) = \frac{P(x)}{Q(x)}$ is called a rational fct.

Question: What is $\int R(x) dx$?

recall example of partial fractions:

$$\frac{1}{x(x-1)} = \frac{a}{x} + \frac{b}{x-1} = \frac{a(x-1)+bx}{x(x-1)} \Rightarrow a+b=0, -a=1$$

$$\Rightarrow a=-1 \Rightarrow b=1$$

$$\Rightarrow \frac{1}{x(x-1)} = \frac{1}{x-1} - \frac{1}{x}$$

$$\Rightarrow \int \frac{1}{x(x-1)} dx = \ln(x-1) - \ln x$$

What about $\frac{3x^3-9x^2+12x-8}{x^2-3x+2}$? $\Rightarrow$ problem with partial fractions since $\deg P > \deg Q$

do the following:

$$\frac{3x^3-9x^2+12x-8-3x(x^2-3x+2)+3x(x^2-3x+2)}{x^2-3x+2}$$

$$= \underbrace{\frac{6x-8}{x^2-3x+2}} + 3x$$

$$= \frac{6x-8}{(x-2)(x-1)} = \frac{a}{x-1} + \frac{b}{x-2} = \frac{a(x-2)+b(x-1)}{(x-2)(x-1)}$$

$$\Rightarrow \begin{array}{l} a+b=6 \\ -2a-b=-8 \end{array} \Rightarrow -a=-2 \Rightarrow a=2,\ b=4$$

$$\Rightarrow \quad \frac{3x^3-9x^2+12x-8}{x^2-3x+2} = \frac{2}{x-1} + \frac{4}{x-2} + 3x$$

$$\Rightarrow \int \frac{3x^3-9x^2+12x-8}{x^2-3x+2}\,dx = 2\ln(x-1) + 4\ln(x-2) + \frac{3}{2}x^2$$

general receipe: if $\deg P > \deg Q$ $(\deg P = n,\ \deg Q = m)$

$$\Rightarrow \frac{P(x)}{Q(x)} = \frac{a_n x^n + \dots}{b_m x^m + \dots} = \frac{a_n x^n - \frac{a_n}{b_m}x^{n-m}Q(x) + \frac{a_n}{b_m}x^{n-m}Q(x) + \dots}{Q(x)}$$

$$= \frac{a_{n-1}x^{n-1} + \dots}{Q(x)} + \frac{a_n}{b_m}x^{n-m}$$

repeat until $\frac{P(x)}{Q(x)} = \frac{\tilde{P}(x)}{Q(x)} + \tilde{\tilde{P}}(x)$ with $\deg \tilde{P} < \deg Q$

in general: any polynomial can be factorized:

$$Q(x) = \underbrace{\prod_{k=1}^{n}}_{\text{product}} (x-\alpha_k)^{m_k}$$

, α_k = roots (possibly complex)

m_k = multiplicity of the k-th root

Thm. (partial fractions): If $\deg P < \deg Q$, then

$$\frac{P(x)}{Q(x)} = \sum_{k=1}^{n} \sum_{j=1}^{m_k} \frac{c_{jk}}{(x-\alpha_k)^j} \quad \text{with some constants } c_{jk}$$

Ex.: $\frac{1}{(x-1)^2}$ cannot be decomposed further into partial fractions

$$\frac{1}{(x-1)^2} = \frac{a}{x-1} + \frac{b}{x-1} = \frac{a(x-1)+b(x-1)}{(x-1)^2} = \frac{a+b}{x-1} \ \lightning$$

further examples: next HW

For integrating $\frac{P(x)}{Q(x)}$: • $j=1 \rightarrow \int \frac{1}{(x-\alpha_k)}\,dx = \ln(x-\alpha_k)$

• $j>1 \rightarrow \int \frac{1}{(x-\alpha_k)^j}\,dx = \frac{(x-\alpha_k)^{-j+1}}{-j+1}$

Summary: how to compute $\int \frac{P(x)}{Q(x)}\,dx$?

→ reduce to $\deg \tilde{P} < \deg Q$

→ factorize Q

→ partial fractions

→ integrate

Remark: • be aware of multiplicity

• a bit ugly for complex roots

↳ alternatively, use factorization

$$Q(x) = \prod_{i=1}^{\ell} \left((x-a_i)^2 + b_i^2 \right)^{m_i} \prod_{i=1}^{k} (x-\gamma_i)^{n_i}$$

with $\gamma_1, \dots, \gamma_k$ the real roots and $a_i \pm i b_i$ the complex roots

explanation of quadratic term:

$$\left(x-(a+ib)\right)\left(x-(a-ib)\right) = (x-a-ib)(x-a+ib)$$

$$= (x-a)^2 - ib(x-a) + ib(x-a) + b^2$$

$$= (x-a)^2 + b^2$$