Session 17
Nov. 12, 2018

- Power Series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \text{ with radius of convergence } \rho$$

we know: $f'(x) = \sum_{k=1}^{\infty} a_k k x^{k-1}$

$$\int f(x)\,dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$$

Note: • Radius of conv. for $\sum a_k x^k$, $\sum a_k k x^{k-1}$, $\sum \frac{a_k}{k+1} x^{k+1}$ is the same because $\lim_{k\to\infty} \sqrt[k]{k} = 1 = \lim_{k\to\infty} \sqrt[k]{\frac{1}{k+1}}$

• If $f(x)$ conv. at $x = \pm\rho$, then $f'(x)$ does not necessarily conv. there

(Ex.: $\sum \frac{1}{k^2} x^k$ conv. at $x=1$, but derivative $\sum \frac{1}{k} x^{k-1}$ does not)

• If $f(x)$ conv. at $x = \pm\rho$, then so does $\int f(x)\,dx$

(the $\frac{1}{k+1}$ makes convergence better)

Ex.: $\int e^{-x^2} dx$ can not be expressed in terms of elementary fct.s but at least we know

$$\int e^{-x^2} dx = \int \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!} dx = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k!(2k+1)}$$

Finally:

write $f(x) = f(a) + \underbrace{\int_a^x f'(t)\,dt}$

int. by parts: $= (t-x)f'(t)\Big|_a^x - \int_a^x (t-x) f''(t)\,dt$

$= (x-a)f'(a) + \int_a^x (x-t) f''(t)\,dt$

repeating leads to

<u>Thm. (Taylor expansion):</u>

Let $f(x)$ be $n+1$ times continously differentiable on $[a,x]$ (or $[x,a]$ for $x<a$), then

$$\boxed{f(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt}$$

Note: this gives us a nicer expression for the rest/remainder term.

3.2 Improper Integrals

integrals when $[a,b]$ is replaced by infinite interval or $f(x)$ not bounded

Def.: If $f:[a,\infty)\to\mathbb{R}$ is integrable on $[a,b]$ for all $b>a$, then

we def. $\boxed{\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx}$.

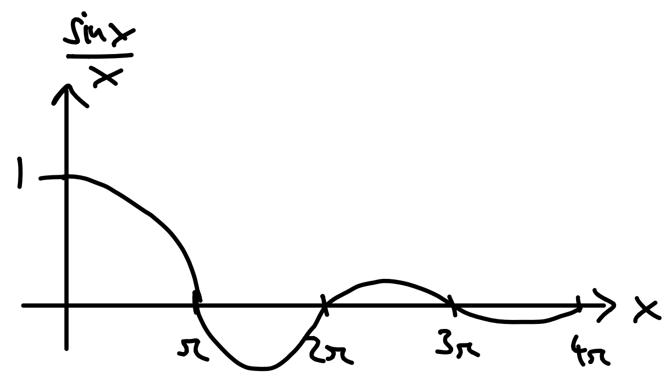
If the limit exists, we say the improper integral $\int_a^\infty f(x)\,dx$ exists.

Ex.: $\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b$

$$= \lim_{b\to\infty}\left[-e^{-b}+1\right] = 1$$

in short: $\int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1$

Ex.: $\int_0^\infty \frac{\sin x}{x}\,dx$



$$\Rightarrow \int_0^\infty \frac{\sin x}{x}\,dx = \sum_{k=0}^\infty (-1)^k \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}\,dx$$

and this series converges due to Leibniz criterion very useful application:

Thm.: let $f(x) \geq 0$ be non-increasing on $[1,\infty)$. Then

$$\sum_{k=1}^{\infty} f(k) \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges}$$

$$\iff \int_a^{\infty} f(x)\,dx \text{ converges (any } a>1 \text{ fixed)}$$

(see homework)

Def.: If $f:\mathbb{R}\to\mathbb{R}$ integrable for all intervals $[a,b]$, then

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx := \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx} \text{ \underline{if both exist}}$$

Ex.: $\int_{-\infty}^{\infty} x\,dx$ does not exist since $\int_0^{\infty} x\,dx$ does not exist

$$\left(\int_{-\infty}^{\infty} x\,dx \neq \lim_{b\to\infty} \int_{-b}^{b} x\,dx = 0 \right)$$

Ex.: $\int_0^{\infty} \frac{1}{x^2+1}\,dx = \arctan x \Big|_0^{\infty} = \frac{\pi}{2}$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \pi$$

Def.: If $f:(a,b]\to\mathbb{R}$ is integrable on intervals $[a+\varepsilon,b]$ for all $0<\varepsilon<b-a$, then $\boxed{\int_a^b f(x)\,dx := \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^{b} f(x)\,dx}$.

$\underline{\text{Ex.:}}$ $\displaystyle\int_0^1 \frac{1}{x^\alpha}\,dx = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 x^{-\alpha}\,dx = \lim_{\varepsilon\to 0^+} \frac{x^{-\alpha+1}}{-\alpha+1}\Bigg|_1^{\varepsilon}$

$(\alpha \neq 1)$

$$= \lim_{\varepsilon\to 0^+}\left[\frac{1}{1-\alpha} - \frac{\varepsilon^{1-\alpha}}{1-\alpha}\right] = \begin{cases} \text{divergent if } \alpha > 1 \\ \dfrac{1}{1-\alpha} \text{ if } 0 < \alpha < 1 \end{cases}$$

note: $\alpha = 1$: $\displaystyle\int_0^1 \frac{1}{x}\,dx = \lim_{\varepsilon\to 0^+} \ln x\Big|_\varepsilon^1 = \lim_{\varepsilon\to 0^+}(-\ln\varepsilon)$ diverges