3.3 Sequences of Functions

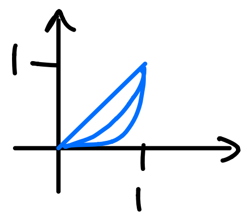
Session 18
Nov. 14, 2018

consider sequence of fct.s $(f_n(x))_{n\in\mathbb{N}}$, $f_n : [a,b] \to \mathbb{R}$

If for each fixed $x \in [a,b]$ $f_n(x) \xrightarrow{n\to\infty} f(x)$, we say f_n converges pointwise to f.

We want to know:

- If f_n is continuous $\forall n$, is f continuous?
- Is $\lim\limits_{n\to\infty} \int f_n(x)dx = \int \lim\limits_{n\to\infty} f_n(x)\,dx = \int f(x)dx$?
- Is $\lim\limits_{n\to\infty} f_n'(x) = f'(x)$?



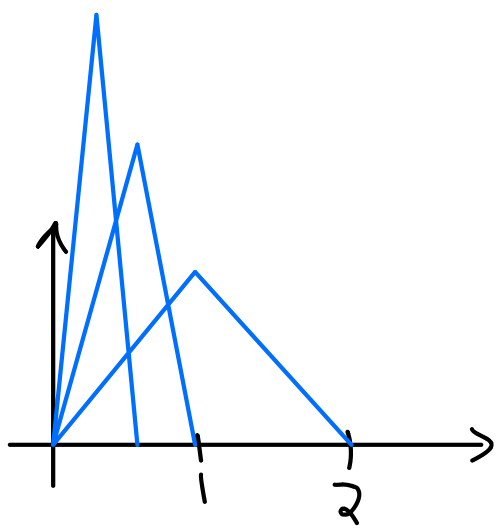
Ex.: $f_n(x) = x^n$, $x \in [0,1]$

each $f_n(x)$ is continuous, but $\lim\limits_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{for } x \in [0,1) \\ 1 & \text{for } x = 1 \end{cases}$

is not continuous.

Ex.:

$$f_n(x) = \begin{cases} n^2 x & , 0 \leq x \leq \frac{1}{n} \\ 2n - n^2 x & , \frac{1}{n} < x < \frac{2}{n} \\ 0 & , \frac{2}{n} \leq x \leq 2 \end{cases}$$



$$\int_0^2 f_n(x)dx = \int_0^{\frac{1}{n}} n^2 x\,dx + \int_{\frac{1}{n}}^{\frac{2}{n}} (2n - n^2 x)dx$$

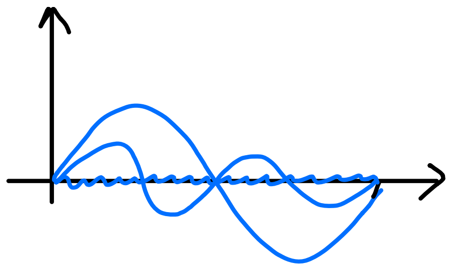
$$= n^2 \frac{x^2}{2}\Big|_0^{\frac{1}{n}} + \dots \quad = \frac{1}{2} + \frac{1}{2} = 1$$

but $\lim_{n\to\infty} f_n(x) = 0 = f(x)$ for all $x \in [0,2]$

So $\lim_{n\to\infty} \int_0^2 f_n(x)\,dx = 1 \neq \int_0^2 f(x)\,dx = 0$

<u>Ex.:</u> $f_n(x) = \frac{1}{n}\sin(nx)$

$\lim_{n\to\infty} f_n(x) = 0 = f(x)$



but $f_n'(x) = \frac{1}{n}\, n\cos(nx) = \cos(nx)$ which doesn't have a limit

so here $\lim_{n\to\infty} f_n'(x)$ doesn't exist although $f'(x) = 0$.

Key concept: uniform convergence

recall def. of convergence: $a_n \xrightarrow{n\to\infty} a$ means a_n becomes arbitrarily close to a for large n

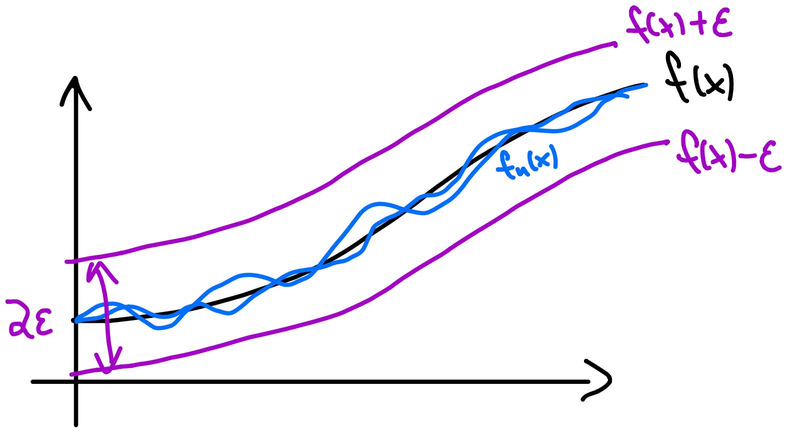
more formally: $\underbrace{\forall \varepsilon > 0}_{\text{arbitrarily close}}\ \underbrace{\exists N \text{ s.t. } \forall n \geq N}_{\text{large } n} : \underbrace{|a_n - a| < \varepsilon}$

for $f_n(x)$ this leads to pointwise conv.:

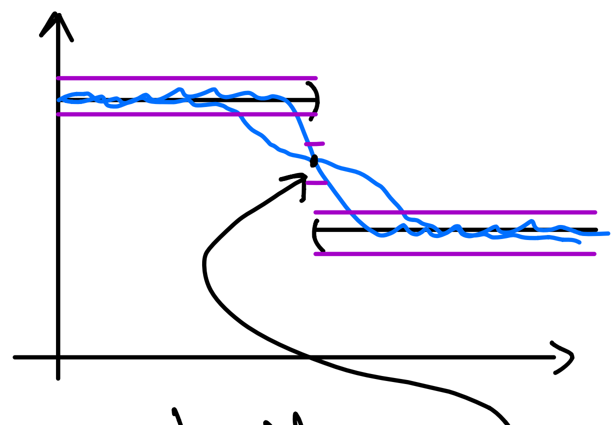
$\underbrace{\forall x \in [a,b]}_{\text{fix } x}\ \underbrace{\forall \varepsilon_x > 0}_{\text{each } x \text{ can have its own } \varepsilon_x}\ \exists N \text{ s.t. } \forall n \geq N : |f_n(x) - f(x)| < \varepsilon_x$

a stronger notion is uniform convergence:

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$ and $\forall x \in [a,b]$: $|f_n(x) - f(x)| < \varepsilon$



uniform convergence



not uniform conv. but pointwise

With that we have the following results (we don't give proofs here):

Thm.: $f_n : [a,b] \to \mathbb{R}$ continuous and $f_n \xrightarrow{n\to\infty} f$ uniformly. Then f is continuous.

Thm.: $f_n : [a,b] \to \mathbb{R}$ integrable and $f_n \xrightarrow{n\to\infty} f$ uniformly. Then f is integrable and $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$.

Thm.: $f_n : (a,b) \to \mathbb{R}$ continuously differentiable, $f_n \xrightarrow{n\to\infty} f$ pointwise, $f_n' \xrightarrow{n\to\infty} g$ uniformly for some g. Then f is continuously differentiable and $\lim_{n\to\infty} f_n'(x) = f'(x)$.