4. Differential Equations

Session 19
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here: we consider ODEs (ordinary differential equations)

these are eq.s involving a fct. $y(x)$ depending on one variable x, and its derivatives

(PDE = partial differential eq.: $y(x_1, \dots, x_n)$, i.e., many variables)

first order: $y'(x) = f(x, y(x))$, i.e., only first derivative involved

n-th order: highest involved derivative $y^{(n)}(x)$

solving ODE means finding $y(x)$ that solves ODE

for $y'(x) = f(x, y(x))$ we specify initial condition $y(x_0) = y_0$

Ex.: $\frac{dy}{dx} = \lambda y$, $\lambda \in \mathbb{R}$ (population growth, $\lambda > 0$ growth, $\lambda < 0$ decay)

solve by separation of variables:

write formally $\frac{dy}{y} = \lambda dx$ and integrate both sides:

$$\int_{y_0}^{y} \frac{d\tilde{y}}{\tilde{y}} = \int_{x_0}^{x} \lambda d\tilde{x} \implies \ln \tilde{y} \Big|_{y_0}^{y} = \lambda \tilde{x} \Big|_{x_0}^{x}$$

$$\implies \ln y - \ln y_0 = \ln \frac{y}{y_0} = \lambda (x - x_0)$$

$$\implies y(x) = y_0 e^{\lambda (x - x_0)}$$

with initial condition $y(x_0) = y_0 e^{\lambda(x_0 - x_0)} = y_0$

alternatively, we could just write

$$\int \frac{dy}{y} = \int \lambda dx \Rightarrow \ln y = \lambda x + C$$

$$\Rightarrow y(x) = \underbrace{e^C}_{A} e^{\lambda x} = A e^{\lambda x}$$

initial condition at x_0: $y(x_0) = A e^{\lambda x_0}$ (A = initial cond. at $x_0 = 0$)

express A in terms of x_0 and y_0: $A = y_0 e^{-\lambda x_0}$

(we could plug that back in and find $y(x) = y_0 e^{\lambda(x - x_0)}$)

4.1 Some Types of Integrable ODEs

integrable = find explicit sol. by integration

- $y'(x) = f(x) g(y)$, called separable ODE

$\Rightarrow$ write formally $\frac{dy}{dx} = f(x) g(y)$ as $\frac{dy}{g(y)} = f(x) dx$ and integrate

$\Rightarrow$ solve $\int \frac{dy}{g(y)} = \int f(x) dx$

(if possible; conditions are, e.g., f and g continuous and g non-zero)

- $y'(x) = f(x)\,y$, called linear homogeneous ODE

as before, $\int \frac{dy}{y} = \int f(x)\,dx$, so $\ln y = \int f(x)dx + C$

$\Rightarrow \; y(x) = A e^{\int f(x)dx}$

- $y'(x) = f(x)y + g(x)$, called linear inhomogeneous ODE

idea: write $y(x) = u(x)\cdot v(x)$

then $\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx} = f u\cdot v + g$

solve $\frac{du}{dx} = fu$ and then $\frac{dv}{dx} = \frac{g}{u}$

we have $u(x) = e^{\int_0^x f(\tilde{x})d\tilde{x}}$ and $v(x) = \int_0^x \frac{g(\tilde{x})}{u(\tilde{x})}d\tilde{x} + C$

$\Rightarrow$ solution $y(x) = e^{\int_0^x f(\tilde{x})d\tilde{x}}\left(C + \int_0^x \frac{g(\tilde{x})}{u(\tilde{x})}d\tilde{x}\right)$

- $y'(x) = f(x)y + g(x)y^n$, called Bernoulli differential eq.

again, write $y(x) = u(x)v(x)$

$\Rightarrow \; u'v + uv' = f\underbrace{uv}_{y} + g\underbrace{u^n v^n}_{y^n}$

$\Rightarrow$ solve $u' = fu$ $\Rightarrow u(x) = e^{\int_0^x f(\tilde{x})d\tilde{x}}$

and $v' = g u^{n-1} v^n$ $\Rightarrow \dfrac{dv}{v^n} = g u^{n-1} dx$

$$\Rightarrow \frac{v^{-n+1}}{-n+1} = \int_0^x g(\tilde{x}) u^{n-1}(\tilde{x}) d\tilde{x} + C$$

$$\Rightarrow \text{solution: } y(x) = u(x)\left(C + (1-n)\int_0^x g(\tilde{x}) u^{n-1}(\tilde{x}) d\tilde{x}\right)^{\frac{1}{1-n}}$$

<u>Ex.:</u> logistic growth: $\dfrac{dy}{dx} = \lambda y \left(1 - \dfrac{y}{k}\right)$

(λ growth rate, k, e.g., environmental carrying capacity)

solve by separation of variables (see HW) or using Bernoulli-formula:

$y' = \lambda y - \frac{\lambda}{k} y^2$ $\quad \left(f(x) = \lambda,\ g(x) = -\frac{\lambda}{k}\right)$

from before we get $u(x) = e^{\lambda x}$

$$\Rightarrow y(x) = e^{\lambda x}\Big(\tilde{C} + \underbrace{\int_0^x \frac{\lambda}{k} e^{\lambda \tilde{x}} d\tilde{x}}_{= \frac{1}{k} e^{\lambda \tilde{x}}\big|_0^x = \frac{e^{\lambda x}}{k} + \tilde{\tilde{C}}}\Big)^{-1}$$

$$\Rightarrow \text{solution is } y(x) = e^{\lambda x}\Big(\underbrace{C}_{\tilde{C} + \tilde{\tilde{C}}} + \frac{e^{\lambda x}}{k}\Big)^{-1}$$

• second order ODEs: $y'' = f(x, y, y')$

special cases:

↳ $f = f(x, y')$ (no y involved)

here we can just introduce new variable $p = y'$

then we need to solve first order ODE $p' = f(x, p)$

↳ $f = f(y, y')$ (no x involved)

(look for $p(y)$ s.t. $y' = p(y)$)

$$y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p' \cdot p \implies \text{need to solve } p'p = f(y, p)$$

(for p as fct. of y)

$\implies$ this gives us $p(y)$, then we still need to solve $y' = p(y)$

Ex.: harmonic oscillator

Newton's second law: force = mass · acceleration: $F = ma$

position $x(t)$, velocity $\frac{dx(t)}{dt} = v(t)$, acceleration $\frac{d^2x(t)}{dt^2} = \frac{dv(t)}{dt} = a(t)$

harmonic oscillator: force $F = -kx$ (e.g. spring)

$\implies$ ODE: $m\frac{d^2x}{dt^2} = -kx$ or $\frac{d^2x}{dt^2} = -\omega^2 x$, $\omega = \sqrt{\frac{k}{m}}$

look at $v = v(x(t))$

$$\Rightarrow \frac{dv}{dt} = \frac{dv}{dx}\underbrace{\frac{dx}{dt}}_{v} = -\omega^2 x \Rightarrow \text{solve } \frac{dv}{dx}\cdot v = -\omega^2 x$$

$$\Rightarrow \text{separation of variables: } v\,dv = -\omega^2 x\,dx$$

$$\Rightarrow v(x)^2 = -\omega^2 x^2 + C_1$$

$$\Rightarrow v(x) = \sqrt{C_1 - \omega^2 x^2} = \frac{dx}{dt}$$

$$\Rightarrow \text{need to solve } \frac{dx}{dt} = \sqrt{C_1 - \omega^2 x^2}$$

$$\Rightarrow \int \frac{dx}{\sqrt{C_1 - \omega^2 x^2}} = \int dt$$

$$\Rightarrow \frac{1}{\sqrt{C_1}} \int \frac{dx}{\sqrt{1 - \frac{\omega^2}{C_1}x^2}} \overset{\frac{\omega x}{\sqrt{C_1}} = y}{=} \frac{1}{\sqrt{C_1}} \frac{\sqrt{C_1}}{\omega} \int \frac{dy}{\sqrt{1-y^2}} = \frac{1}{\omega}\int \frac{dy}{\sqrt{1-y^2}}$$

$$= \frac{1}{\omega} \arcsin y = \frac{1}{\omega} \arcsin\left(\frac{\omega}{\sqrt{C_1}} x\right)$$

$$\Rightarrow \frac{1}{\omega} \arcsin\left(\frac{\omega}{\sqrt{C_1}} x\right) = t + C_2$$

$$\Rightarrow \text{solution } x(t) = \frac{\sqrt{C_1}}{\omega} \sin(\omega t + \omega C_2)$$

$$\text{or nicer: } x(t) = \underbrace{A_0}_{\text{amplitude}} \sin(\omega t + \underbrace{\varphi}_{\text{initial phase}})$$

initial position at $t=0$: $x(0) = A_0 \sin\varphi = x_0$

initial velocity at $t=0$: $v(0) = A_0 \omega \cos\varphi = v_0$