4.2 linear ODEs

Session 20
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homogeneous linear ODE of order n:

$$y^{(n)} + a_{n-1}(x) y^{(n-1)} + \dots + a_1(x) y' + a_0(x) y = \begin{cases} 0, & \text{homogeneous} \\ f(x), & \text{inhomogeneous} \end{cases}$$

for some given fct.s $a_0(x), \dots, a_{n-1}(x)$ and $f(x)$

notation: def. differential op.: $\mathcal{L}(y) = y^{(n)} + a_{n-1}(x) y^{(n-1)} + \dots + a_0(x) y$

$$\Rightarrow \mathcal{L}(y) = \begin{cases} 0, & \text{hom.} \\ f(x), & \text{inhom.} \end{cases}$$

linearity: $\mathcal{L}(c_1 y_1 + c_2 y_2) = c_1 \mathcal{L}(y_1) + c_2 \mathcal{L}(y_2)$

so we conclude: If $y_1(x), y_2(x), \dots, y_n(x)$ are sol.s to homogeneous eq., then

$c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x)$ is also a solution

note: $\sum_{j=1}^{n} c_j y_j(x)$ is the general sol. (n constants to be specified by initial conditions) if $y_1(x), \dots, y_n(x)$ are linearly independent, i.e.,

$\sum_{j=1}^{n} c_j y_j(x) = 0$ implies all $c_1, \dots, c_n = 0$

(linearly dependent: can write $y_i(x) = \sum_{\substack{j=1 \\ j \neq i}}^{n} c_j y_j(x)$ for some i)

Ex.:
- $y_1(x) = x^2$ and $y_2(x) = 3x^2$ are lin. dep.: $y_2 = 3y_1$
- $y_1(x) = 1$ and $y_2(x) = x$ are lin. indep.: if $c_1 \cdot 1 + c_2 x = 0 \; \forall x$
  $\Rightarrow c_1 = 0 = c_2$

Lemma: let $y_{gen}(x)$ be the general solution (meaning it has n constants) to the homogeneous eq., and $y_{part}(x)$ one particular solution (possibly without any constants) to the inhomogeneous eq., then $y_{gen} + y_{part}$ is the general solution to the inhomogeneous eq.

Proof: $$\mathcal{L}(y_{gen} + y_{part}) = \underbrace{\mathcal{L}(y_{gen})}_{=0} + \underbrace{\mathcal{L}(y_{part})}_{=f(x)} = f(x)$$

so $y_{gen} + y_{part}$ is sol. to inhom. eq. with right number of constants.

Suppose $\tilde{y}$ is a different sol. to inhom. eq.: $\mathcal{L}(\tilde{y}) = f$

$$\mathcal{L}(\tilde{y} - y_{part}) = \underbrace{\mathcal{L}(\tilde{y})}_{=f(x)} - \underbrace{\mathcal{L}(y_{part})}_{=f(x)} = 0$$

$$\Rightarrow \tilde{y} = \underbrace{y_{part}}_{\text{previous inhom. sol.}} + \underbrace{(\tilde{y} - y_{part})}_{\text{sol. to hom. eq., so already covered by } y_{gen}}$$

from now on we discuss constant a_i's: $y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = \begin{cases} 0 \\ f(x) \end{cases}$

consider hom. case first:

ansatz: $y(x) = e^{\lambda x} \Rightarrow y'(x) = \lambda e^{\lambda x}$, so $y^{(n)}(x) = \lambda^n e^{\lambda x}$

$$\Rightarrow \text{eq. becomes } \underbrace{\left(\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1 \lambda + a_0\right)}_{\chi(\lambda)} e^{\lambda x} = 0$$

so $e^{\lambda x}$ is a sol. if λ is a zero of polynomial $\chi(\lambda)$

If all roots $\lambda_1, \ldots, \lambda_n$ are different we're done:

the gen. sol. is $y(x) = \sum_{i=1}^{n} c_i e^{\lambda_i x}$ (note: $e^{\lambda_i x}$ for different λ_i's are indeed lin. indep.)

Multiple roots:

Ex.: $y^{(n)} = 0 \Rightarrow \chi(\lambda) = \lambda^n$, so root is zero with multiplicity n

gen. sol. is $y(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_{n-1} x^{n-1}$

Ex.: $y''' - 3a y'' + 3a^2 y' - a^3 y = 0$

$\Rightarrow \chi(\lambda) = \lambda^3 - 3a\lambda^2 + 3a^2\lambda - a^3 = (\lambda - a)^3$

here we def. differential op. D_a by $D_a y = y' - ay$

so diff. eq. becomes $D_a^3 y = 0$

note: in gen. • $D_a D_b y = D_a(y' - by) = (y' - by)' - a(y' - by)$

$(= D_a y' - b D_a y) \quad = y'' - by' - ay' + aby$

$= y'' - (a+b)y' + aby$

• $D_b D_a y = D_b(y' - ay) = (y' - ay)' - b(y' - ay)$

$= y'' - (a+b)y' + aby$

$\Rightarrow D_a D_b = D_b D_a$ (D_a and D_b commute)

If $D_a y = 0$ for $y = e^{\lambda x}$ then $\chi(\lambda) = (\lambda - a)$

back to example: ansatz $y(x) = e^{ax} u(x)$

$$\Rightarrow D_a y = y' - ay = ae^{ax}u(x) + e^{ax}u'(x) - a\underbrace{e^{ax}u(x)}_{y(x)} = e^{ax}u'(x)$$

$$D_a^2 y = e^{ax} u''(x)$$

$$D_a^3 y = e^{ax} u'''(x)$$

eq. $D_a^3 y = 0$, so need $u'''(x) = 0$, so $u(x) = c_0 + c_1 x + c_2 x^2$

$\Rightarrow$ so general sol. is $y(x) = e^{ax}(c_0 + c_1 x + c_2 x^2)$

General case:

<u>Thm. (Euler):</u>

Let $\chi(\lambda) = (\lambda - \lambda_1)^{m_1} \cdot (\lambda - \lambda_2)^{m_2} \cdot \ldots \cdot (\lambda - \lambda_k)^{m_k}$

(with $\lambda_1, \ldots, \lambda_k$ distinct). Then gen. sol. is $y(x) = \sum_{i=1}^{k} p_i(x) e^{\lambda_i x}$,

where $p_i(x) = \sum_{j=0}^{m_i - 1} c_{ij} x^j$ (polynomial of degree $m_i - 1$).

<u>Proof:</u> need to solve $D_{\lambda_1}^{m_1} D_{\lambda_2}^{m_2} \cdots D_{\lambda_k}^{m_k} y = 0$

all $D_{\lambda_i}^{m_i}$ commute, so each $p_i(x) e^{\lambda_i x}$ is sol., since

$$D_{\lambda_1}^{m_1} \cdots D_{\lambda_k}^{m_k} \underbrace{D_{\lambda_i}^{m_i}\left(p_i(x) e^{\lambda_i x}\right)}_{=0} = 0.$$

Also, the specified gen. sol. has right number of constants $n = \sum_{i=1}^{k} m_i$.