Session 21
Nov. 26, 2018

note: Euler's formula is still valid if some of the λ_i's are complex. Often: want real solutions.

$\Rightarrow$ Use that if $\lambda_1 = \alpha + i\beta$ $(\alpha, \beta \in \mathbb{R})$ then $\lambda_2 = \overline{\lambda_1} = \alpha - i\beta$ is also a zero

replace $c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$ (constants c_1, c_2) by $C e^{\alpha x} \sin(\beta x + \varphi)$ (constants C, φ) (see HW10, Problem 1)

$\Rightarrow$ gen. real solution

<u>Inhomogeneous Eq.:</u>

find one solution to $\underbrace{y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y}_{\mathcal{L}(y)} = f(x)$

note: if $f = d_1 f_1 + d_2 f_2$, then look at f_1, f_2 separately:

$\mathcal{L}(y_1) = f_1$, $\mathcal{L}(y_2) = f_2$

$\Rightarrow \mathcal{L}(d_1 y_1 + d_2 y_2) = d_1 \mathcal{L}(y_1) + d_2 \mathcal{L}(y_2) = d_1 f_1 + d_2 f_2$

heuristic strategy: if $f(x) = x^j, e^{\alpha x}, e^{\alpha x} \sin(\omega x), \ldots$, then look for sol. that involves similar fct.s

Ex.: $y'' - y' + y = \sin(2x)$

ansatz: $y_{part}(x) = a\sin(2x) + b\cos(2x)$

determine a,b from eq.:

$$y_{part}'' - y_{part}' + y_{part} = (a\sin 2x + b\cos 2x)'' - (a\sin 2x + b\cos 2x)' + (a\sin 2x + b\cos 2x)$$

$$= (-4a + 2b + a)\sin 2x + (-4b - 2a + b)\cos 2x$$

$$= \sin 2x$$

$$\Rightarrow -3a + 2b = 1$$
$$-2a - 3b = 0 \Rightarrow a = -\frac{3}{13},\ b = \frac{2}{13}$$

$\Rightarrow y_{part}(x) = -\frac{3}{13}\sin 2x + \frac{2}{13}\cos 2x$ is one solution to this inhom. eq.

note that also $y_{hom}(x) = Ce^{\frac{1}{2}x}\sin\left(\sqrt{\frac{3}{4}}x + \varphi\right)$

$\Rightarrow$ gen. real sol. $y(x) = Ce^{\frac{1}{2}x}\sin\left(\sqrt{\frac{3}{4}}x + \varphi\right) - \frac{3}{13}\sin 2x + \frac{2}{13}\cos 2x$

for constants C, φ determined by initial conditions

Ex.: $y'(x) + 2y(x) = x + 2$

$y_{part}(x) = ax + b$

$\Rightarrow a + 2(ax+b) = x + 2 \Rightarrow 2a = 1, \; a + 2b = 2 \Rightarrow a = \frac{1}{2}, \; b = \frac{3}{4}$

$\Rightarrow y_{part}(x) = \frac{1}{2}x + \frac{3}{4}$

4.3 Qualitative Properties of Solutions

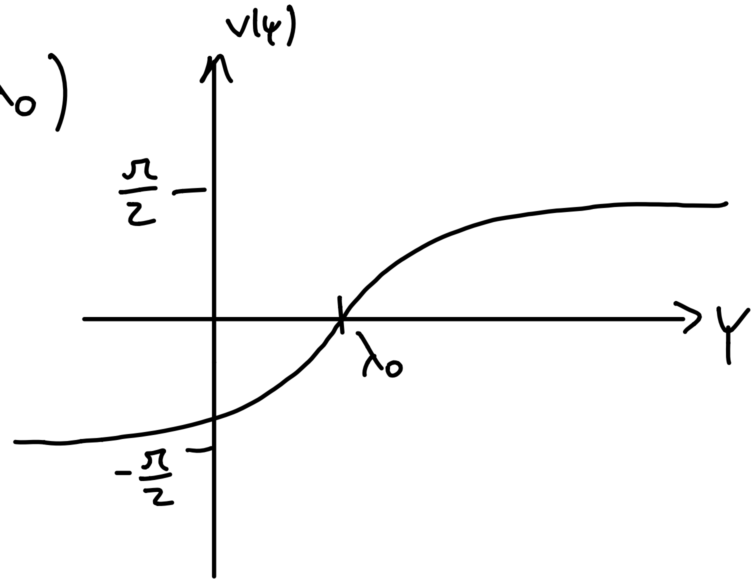
consider autonomous eq.s: $\frac{dy}{dx} = v(y)$, v called vector-field

by separation of variables: $x - x_0 = \int_{y_0}^{y} \frac{1}{v(\tilde{y})} d\tilde{y}$

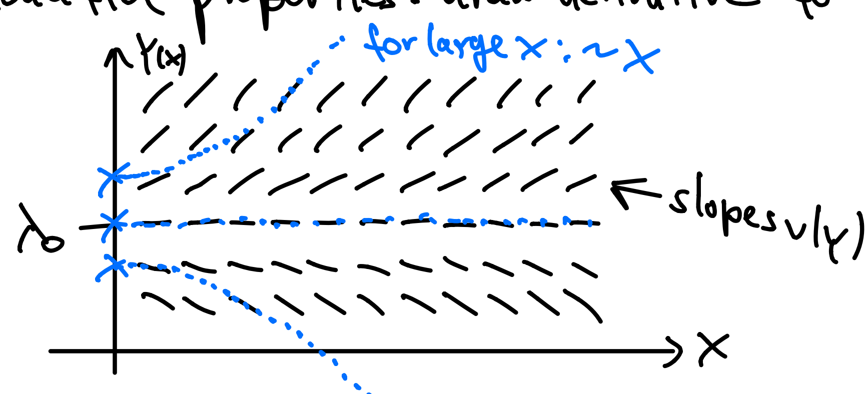
(rigorously: v continuous and non-zero)

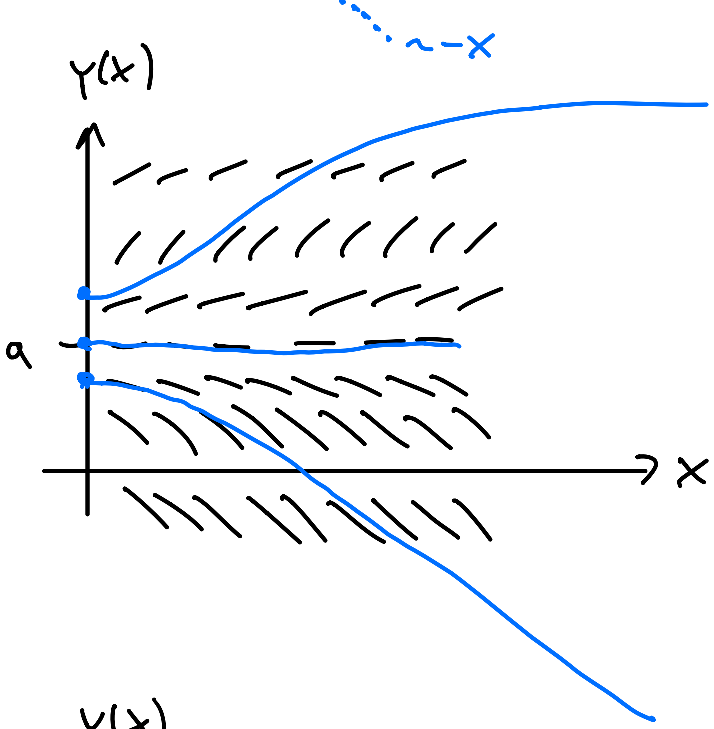
What does the solution do for different initial cond.s?

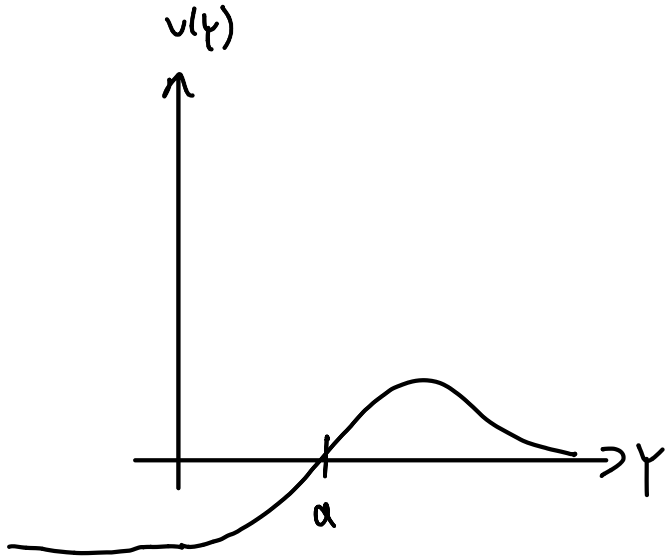
Ex.: $v(y) = \arctan(y - \lambda_0)$

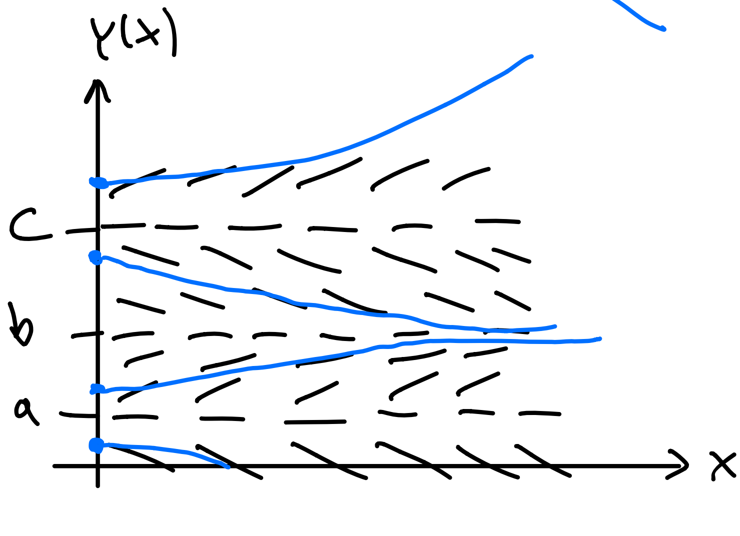


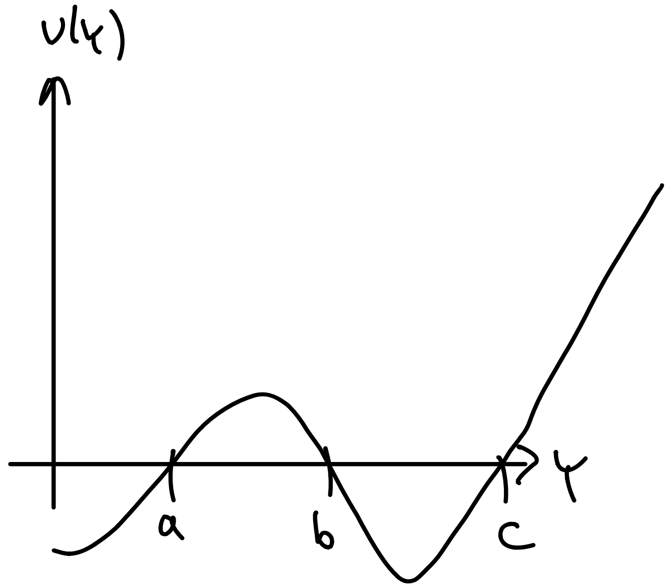
Qualitive properties: draw derivative for different y:











zeros of v: equilibrium positions

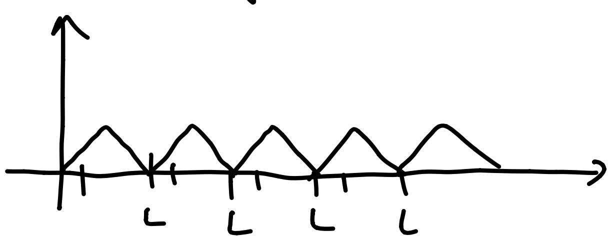
See Bonus Problem of HW 10

5. Fourier Series

Taylor series → good for infinitely often differentiable fct.s

Fourier series → good for some non-continuous fct.s
→ for periodic fct.s (want: sin, cos)

__Def.:__ A fct. F with property $F(x+L) = F(x) \; \forall x$
for some given $L > 0$ is called L-periodic fct.



here: consider 2π periodic fct.s f

if F is L-periodic, then $f(x) = F\left(\frac{L}{2\pi} x\right)$ is 2π periodic, since

$$f(x+2\pi) = F\left(\frac{L}{2\pi}(x+2\pi)\right) = F\left(\frac{L}{2\pi}x + L\right) = F\left(\frac{L}{2\pi}x\right) = f(x)$$

__Def.:__ A fct. of the form $f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos(kx) + b_k \sin(kx)\right)$
with $a_k, b_k \in \mathbb{R}$, is called trigonometric polynomial.

gen. idea: approximate fct.s by trig. polynomials = Fourier series first: given any trig. pol. f, what are a_k, b_k?

fix $\ell \in \mathbb{N}$, then integrate

$$\int_0^{2\pi} f(x)\cos(\ell x)\,dx = \int_0^{2\pi} \frac{a_0}{2}\cos(\ell x)\,dx + \sum_{k=1}^{n} a_k \int_0^{2\pi} \cos(kx)\cos(\ell x)\,dx$$

$$+ \sum_{k=1}^{n} b_k \int_0^{2\pi} \sin(kx)\cos(\ell x)\,dx$$