Session 22
Nov. 28, 2018

first: given any trig. pol. f, what are a_k, b_k?

fix $\ell \in \mathbb{N}$, then integrate

$$\int_0^{2\pi} f(x)\cos(\ell x)\,dx = \int_0^{2\pi} \frac{a_0}{2}\cos(\ell x)\,dx + \sum_{k=1}^{n} a_k \int_0^{2\pi} \cos(kx)\cos(\ell x)\,dx$$
$$+ \sum_{k=1}^{n} b_k \int_0^{2\pi} \sin(kx)\cos(\ell x)\,dx$$

evaluate the integrals using trigonometric identities from

$$e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$
$$= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

we find (see HW10, Problem 4):

$$\int_0^{2\pi} \sin kx \cos \ell x\,dx = 0 \quad \forall k,\ell \in \mathbb{N}$$

$$\int_0^{2\pi} \cos kx \cos \ell x\,dx = 0 = \int_0^{2\pi} \sin kx \sin \ell x\,dx \quad \forall k \neq \ell$$

$$\int_0^{2\pi} \cos^2 \ell x\,dx = \pi = \int_0^{2\pi} \sin^2 \ell x\,dx \quad \forall \ell \geq 1$$

$$\left(\ell = 0: \int_0^{2\pi} \cos(0\cdot x)\,dx = 2\pi\right)$$

$$\Rightarrow a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)\,dx \quad \forall k \geq 0$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(kx)\,dx \quad \forall k \geq 1 \quad (\text{same trick with } \sin(\ell x))$$

complex notation: $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$, $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\left[\frac{a_k}{2}e^{ikx} + \frac{a_k}{2}e^{-ikx} - \frac{ib_k}{2}e^{ikx} + \frac{ib_k}{2}e^{-ikx}\right]$$

$$= \sum_{k=-n}^{n} c_k e^{ikx}$$

with $c_k = \frac{1}{2}(a_k - ib_k)$, $k>0$

$c_{-k} = \frac{1}{2}(a_k + ib_k)$, $k>0$

$c_0 = \frac{a_0}{2}$

note: $\int (a(x)+ib(x))dx = \int a(x)dx + i\int b(x)dx$

(a, b real fct.s)

$$\Rightarrow \int_0^{2\pi} e^{ikx}dx = \begin{cases} 2\pi, & k=0 \\ \frac{1}{ik}e^{ikx}\Big|_0^{2\pi} = \frac{1}{ik}\left(e^{ik2\pi} - e^0\right) = 0, & k\neq 0 \end{cases}$$

similar to before:
(fix $-n \le \ell \le n$)

$$\int_0^{2\pi} f(x)e^{-i\ell x}dx = \sum_{k=-n}^{n} c_k \underbrace{\int_0^{2\pi} e^{i(k-\ell)x}dx}_{\substack{=0 \text{ unless } k=\ell \\ \text{for } k=\ell \text{ int.}=2\pi}} = 2\pi c_\ell$$

$$\Rightarrow c_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx$$

$c_k = \hat{f}(k)$ are called Fourier coefficients

$$\mathcal{F}[f](x) = \lim_{n\to\infty} \underbrace{\sum_{k=-n}^{n} \hat{f}(k)e^{ikx}}_{\mathcal{F}_n[f](x)} = \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx}$$ is called Fourier series

(well-defined for any Riemann-integrable f)

Main questions:

- convergence and in which sense?
- does $\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}$ converge to f?

first observation: suppose $f(x) = \sum_{k=-\infty}^{\infty} \gamma_k e^{ikx}$ converges uniformly (for some given γ_k)

$$\Rightarrow \hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \sum_{\ell=-\infty}^{\infty} \gamma_\ell e^{i\ell x} e^{-ikx} dx$$

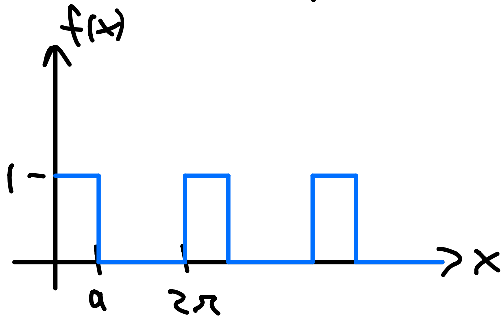
uniform conv.

$$\overset{\curvearrowright}{=} \frac{1}{2\pi} \sum_{\ell=-\infty}^{\infty} \gamma_\ell \underbrace{\int_0^{2\pi} e^{i(\ell-k)x} dx}_{= \begin{cases} 0 & \text{for } k \neq \ell \\ 2\pi & \text{for } k = \ell \end{cases}} = \gamma_k$$

for uniform conv. we have that f equals its Fourier series

but: in general we have neither uniform nor pointwise convergence

<u>Ex.:</u> $f(x) = \begin{cases} 1 & \text{for } 0 \leq x < a \\ 0 & \text{for } a \leq x < 2\pi \end{cases}$



step fct.

here: $c_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi} \int_0^a dx = \frac{a}{2\pi}$

$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx}\,dx = \frac{1}{2\pi} \int_0^a e^{-ikx}\,dx = \frac{1}{2\pi} \frac{1}{(-ik)} e^{-ikx}\Big|_0^a$

$= \frac{i}{2\pi k} \left(e^{-ika} - 1\right) \quad , k \neq 0$

or: $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx = \frac{a}{\pi}$

$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(kx)\,dx = \frac{1}{\pi} \int_0^a \cos(kx)\,dx = \frac{1}{\pi k} \sin kx \Big|_0^a$

$= \frac{\sin(ka)}{\pi k} \quad , k \neq 0$

$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(kx)\,dx = \frac{1}{\pi} \int_0^a \sin(kx) = \frac{-1}{\pi k} \underbrace{\cos(kx)\Big|_0^a}_{= \cos(ka) - 1}$

$= \frac{1 - \cos(ka)}{\pi k} \quad , k \neq 0$

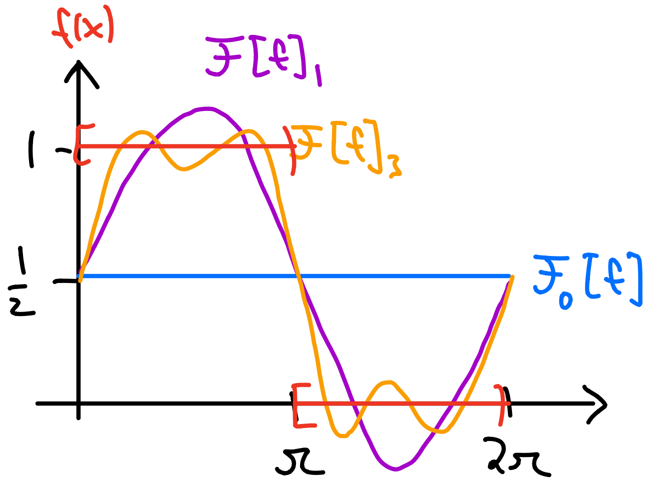
for ex.: $a = \pi \implies a_0 = 1$

$a_k = 0 \quad \forall k \geq 1$

$b_k = \frac{1 - (-1)^k}{\pi k} \quad \forall k \geq 1$

$\implies \mathcal{F}[f](x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(1 - (-1)^k)}{\pi k} \sin kx$

$= \frac{1}{2} + \sum_{k \text{ odd}} \frac{2}{\pi k} \sin kx$



$$\mathcal{F}[f](0) = \frac{1}{2} = \mathcal{F}[f](2\pi)$$

$$\mathcal{F}[f](\pi) = \frac{1}{2}$$

$\Rightarrow$ neither uniform nor pointwise convergence!

need a type of convergence that neglects when fct.s are different at a few points

one good choice: $\int_0^{2\pi} |f(x) - g(x)|^p \, dx$, here: $p=2$ is good