Session 23
Dec. 3, 2018

notation: we introduce $\boxed{\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} \overline{f(x)}\,g(x)\,dx}$

($\overline{f}$ = complex conjugate of $f$)

analogous to $\mathbb{R}^3$, we call this a scalar product

note: • "length" of $f$ is $\boxed{\|f\| = \sqrt{\langle f, f\rangle}}$, called (semi-)norm

• "distance" between $f$ and $g$ is $d(f,g) = \|f-g\|$, called metric

• recall: Cauchy-Schwarz inequality $|\langle f, g\rangle| \leq \|f\|\cdot\|g\|$

$$\left(\|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,dx\right)$$

• also: $\|f+g\| = \sqrt{\langle f+g, f+g\rangle}$

$$= \sqrt{\underbrace{\langle f,f\rangle}_{\|f\|^2} + \underbrace{\langle f,g\rangle + \langle g,f\rangle}_{\leq 2\|f\|\cdot\|g\|} + \underbrace{\langle g,g\rangle}_{\|g\|^2}}$$

$$\leq \sqrt{(\|f\|+\|g\|)^2}$$

$= \|f\| + \|g\|$, called triangle inequality

• define "basis" functions $e_k(x) = e^{ikx}$

these are orthonormal:

"Kronecker delta"

$$\langle e_k, e_\ell\rangle = \frac{1}{2\pi}\int_0^{2\pi} \overline{e_k(x)}\,e_\ell(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{-ix(k-\ell)}\,dx = \delta_{k\ell} := \begin{cases} 1, & k=\ell \\ 0, & k\neq\ell \end{cases}$$

then $c_k = \hat{f}(k) = \langle e_k, f \rangle$

$$\Rightarrow \mathcal{F}[f](x) = \sum_{k=-\infty}^{\infty} \underbrace{\langle e_k, f \rangle}_{\in \mathbb{C}} e_k$$ (any vector = linear combination of basis vectors)

<u>Def.:</u> If $\|f_n - f\| \xrightarrow{n\to\infty} 0$, we say $f_n$ converges to $f$ in quadratic mean.

<u>lemma:</u> Consider fct. $f$ with Fourier coefficient $c_k$. Then $\forall n \in \mathbb{N}$ we have

$$\|f - \sum_{k=-n}^{n} c_k e_k\|^2 = \|f\|^2 - \sum_{k=-n}^{n} |c_k|^2$$

Consequences: • $\sum_{k=-\infty}^{\infty} |c_k|^2 \le \|f\|^2 = \frac{1}{2\pi} \int |f(x)|^2 dx$ (Bessel inequality)

• $\mathcal{F}_n[f] \xrightarrow{n\to\infty} f$ in quadratic mean

$$\Longleftrightarrow \sum_{k=-\infty}^{\infty} |c_k|^2 = \|f\|^2 \text{ (completeness relation)}$$

<u>Proof:</u> $\|f - \underbrace{\sum_{k=-n}^{n} c_k e_k}_{=g}\|^2 = \|f-g\|^2 = \langle f-g, f-g \rangle$

$= \|f\|^2 + \|g\|^2 - \langle f, g \rangle - \langle g, f \rangle$

$\hookrightarrow \langle f, g \rangle = \sum_{k=-n}^{n} c_k \underbrace{\langle f, e_k \rangle}_{\overline{c_k}} = \sum_{k=-n}^{n} |c_k|^2$

$\hookrightarrow \|g\|^2 = \sum_{k=-n}^{n} c_k \sum_{j=-n}^{n} \overline{c_j} \underbrace{\langle e_j, e_k \rangle}_{=\delta_{k\ell}} = \sum_{k=-n}^{n} |c_k|^2$

$$\Rightarrow \|f-g\|^2 = \|f\|^2 + \sum |c_k|^2 - \sum |c_k|^2 - \sum |c_k|^2$$
$$= \|f\|^2 - \sum_{k=-n}^{n} |c_k|^2$$

next: sketch of proof that for all Riemann integrable $2\pi$-periodic fct.s

we have $\mathcal{F}_n[f] \xrightarrow{n\to\infty} f$ in quadratic mean

strategy: • show convergence for step fct.s

• enclose $f$ from above and below by step fct.s

<u>Lemma</u>: For $f(x) = \begin{cases} 1, & 0 \le x < a \\ 0, & a \le x < 2\pi \end{cases}$, $a \in [0, 2\pi]$ fixed,

we have $\mathcal{F}_n[f] \xrightarrow{n\to\infty} f$ in quadratic mean

<u>Proof:</u> we show $\sum_{k=-\infty}^{\infty} |c_k|^2 = \|f\|^2$

from example: $c_k = \frac{i}{2\pi k}\left(e^{-ika} - 1\right)$, $c_0 = \frac{a}{2\pi}$

$$\Rightarrow |c_k|^2 = \frac{1}{4\pi^2 k^2}\left(e^{-ika} - 1\right)\left(e^{ika} - 1\right)$$
$$= \frac{1}{4\pi^2 k^2}\left(2 - \underbrace{\left[e^{-ika} + e^{iak}\right]}_{2\cos(ka)}\right) = \frac{1 - \cos(ka)}{2\pi^2 k^2}$$

now: $\sum_{k=-\infty}^{\infty} |c_k|^2 = \underbrace{\frac{a^2}{4\pi^2}}_{|c_0|^2} + \sum_{k=1}^{\infty} \frac{1}{\pi^2 k^2} - \sum_{k=1}^{\infty} \frac{\cos(ka)}{\pi^2 k^2}$

next: calculate $F(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$ (conv. uniformly)

summary of computation: use $F'(x) = -\sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$

then compute partial sums: $\sum_{k=1}^{n} \frac{\sin(kx)}{k} = \int_{\pi}^{x} \underbrace{\sum_{k=1}^{n} \cos(kt)}\, dt$

evaluate this using Dirichlet
kernel from HW4, Problem 6

... justify exchanges of limits...

$\Rightarrow F(x) = \left(\frac{x-\pi}{2}\right)^2 - \frac{\pi^2}{12} \quad \forall x \in [0, 2\pi)$

in particular: $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6}$

next: $\sum_{k=-\infty}^{\infty} |c_k|^2 = \ldots = \frac{a}{2\pi} = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx = \|f\|^2$