Session 24
Dec. 5, 2018

note: Dirichlet kernel $D_n(x) = \sum_{k=-n}^{n} e^{ikx}$

see HW $\overset{\text{}}{=} \dfrac{\sin((n+\frac{1}{2})x)}{\sin(\frac{x}{2})}$

$$= 1 + 2\sum_{k=1}^{n} \cos(kx)$$

note: $\mathcal{F}_n[f](x) = \sum_{k=-n}^{n} \hat{f}(k) e^{ikx}$

$$= \sum_{k=-n}^{n} \frac{1}{2\pi} \int_0^{2\pi} f(y) e^{-iky} dy \, e^{ikx}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \underbrace{\sum_{k=-n}^{n} e^{ik(x-y)}}_{= D_n(x-y)} f(y) \, dy$$

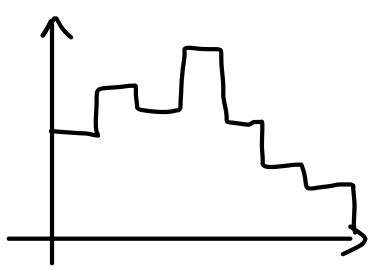
$$= \frac{1}{2\pi} \int_0^{2\pi} D_n(x-y) f(y) \, dy$$

$$=: (D_n * f)(x) \text{, called convolution}$$

summary of next steps:

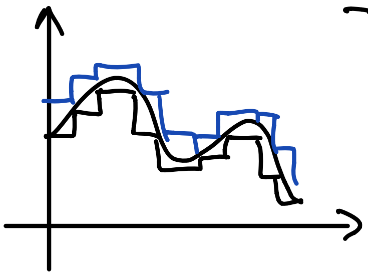
Lemma: If f is a linear combination of step fct.s, then still

$$\|f - \mathcal{F}_n[f]\| \xrightarrow{n\to\infty} 0.$$

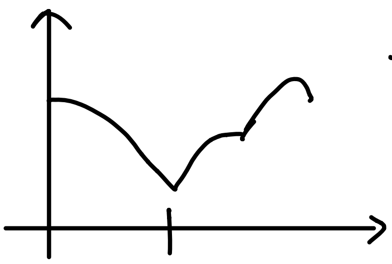


Thm.: Let $f:\mathbb{R}\to\mathbb{C}$ be 2π-periodic and Riemann-integrable on $[0,2\pi]$.

Then $\|f-\mathcal{F}_n[f]\|\xrightarrow{n\to\infty}0$.



Lemma: Let $f:\mathbb{R}\to\mathbb{C}$ be 2π-periodic, continuous and piecewise continuously differentiable (there are $0=t_0<t_1<\dots<t_{n-1}<t_n=2\pi$ s.t. $f(x)$ is cont. differentiable on $[t_k,t_{k+1}]$).



Then $\mathcal{F}_n[f]\xrightarrow{n\to\infty}f$ uniformly.

Proof: omitted here. Strategy: compute c_k's, use integration by parts to gain $\frac{1}{k}$ factor $\Rightarrow$ uniform conv. of $\mathcal{F}_n[f]$ $\Rightarrow$ conv. is to f.

Ex.: Fresnel integrals: $\int_0^\infty \cos(x^2)\,dx$, $\int_0^\infty \sin(x^2)\,dx$

in HW we showed they exist

One can use Fourier series and uniform conv. of $e^{i\frac{x^2}{2\pi}}$ to show that

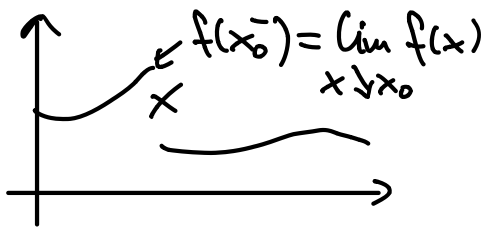
$$\int_{-\infty}^{\infty} e^{i\frac{x^2}{2\pi}}\,dx=\pi(1+i)\overset{x=\sqrt{2\pi}\,y}{=}\int_{-\infty}^{\infty}\underbrace{e^{iy^2}}_{=\cos y^2+i\sin y^2}\sqrt{2\pi}\,dy$$

$$\Rightarrow \int_0^\infty \cos y^2\,dy=\frac{1}{2}\int_{-\infty}^{\infty}\cos y^2\,dy=\frac{1}{2}\frac{\pi}{\sqrt{2\pi}}=\frac{1}{2}\sqrt{\frac{\pi}{2}}=\int_0^\infty \sin y^2\,dy$$

Some general properties of Fourier series:

- f piecewise continuous, and piecewise differentiable; then at points of discontinuity x_0 we have

$$\mathcal{F}_n[f](x_0) \xrightarrow{n\to\infty} \frac{\lim_{x\downarrow x_0} f(x) + \lim_{x\uparrow x_0} f(x)}{2}$$

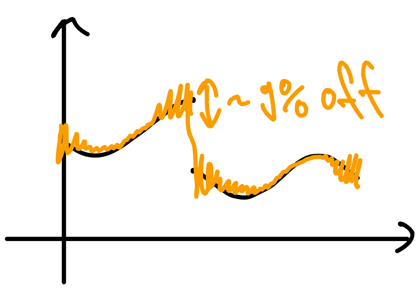


- f piecewise continuous, and piecewise differentiable, say discontinuity at x_0 with gap $g = f(x_0^+) - f(x_0^-)$, then

$$\mathcal{F}_n[f]\left(x_0 + \frac{\pi}{n}\right) \xrightarrow{n\to\infty} f(x_0^+) + g\cdot c, \text{ with } c \approx 0.089\ldots$$

$$\mathcal{F}_n[f]\left(x_0 - \frac{\pi}{n}\right) \xrightarrow{n\to\infty} f(x_0^-) - g\cdot c$$

this is called Gibbs phenomenon



Summary:

Fourier series: $\mathcal{F}[f](x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$, $c_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx$

quadratic mean / norm: $\|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx$

always: Bessel inequality: $\sum_{k=-\infty}^{\infty} |c_k|^2 \leq \|f\|^2$

also $\|\underbrace{\mathcal{F}_n[f]}_{= \sum_{k=-n}^{n} c_k e^{ikx}} - f\| \xrightarrow{n\to\infty} 0 \iff \sum_{k=-\infty}^{\infty} |c_k|^2 = \|f\|^2$

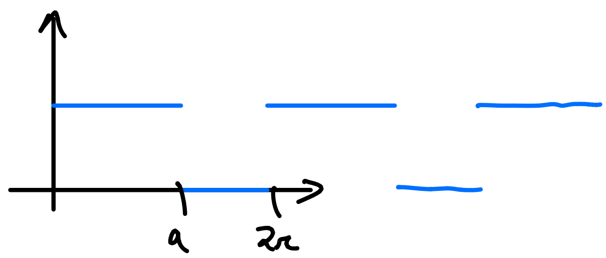
(completeness relation)

Theorems: • f Riemann-integrable $\Rightarrow \|\mathcal{F}_n[f] - f\| \xrightarrow{n\to\infty} 0$

• f cont. and piecewise cont. differentiable $\Rightarrow \mathcal{F}_n[f] \xrightarrow{n\to\infty} f$ uniformly

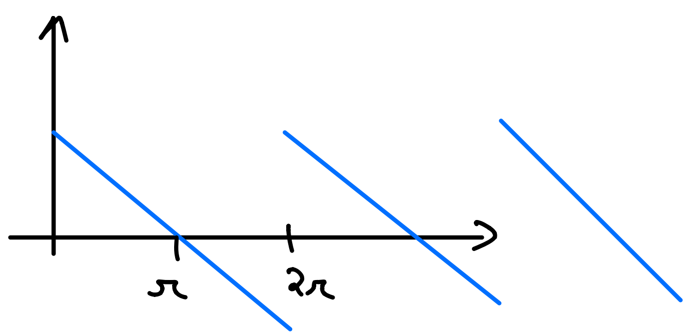
Examples of Fourier series:

– $f(x) = \begin{cases} 1 , & 0 \leq x < a \\ 0 , & a \leq x < 2\pi \end{cases}$



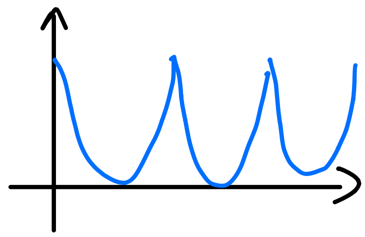
$$\mathcal{F}[f](x) = \frac{a}{2\pi} + \sum_{\substack{k=-\infty \\ (k\neq 0)}}^{\infty} \frac{i}{2\pi k} (e^{-ika} - 1) e^{ikx}$$

– $f(x) = -\frac{(x-\pi)}{2}$ for $x \in [0, 2\pi]$



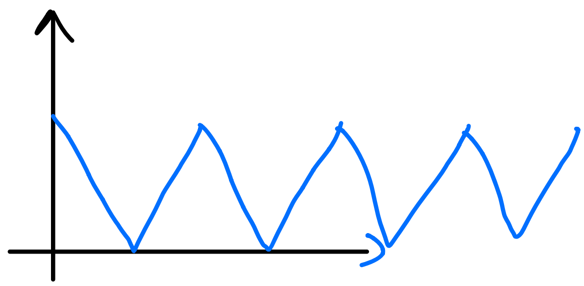
$$\mathcal{F}[f](x) = \sum_{\substack{k=-\infty \\ (k\neq 0)}}^{\infty} \frac{(-i)}{2k} e^{ikx} = \sum_{k=1}^{\infty} \frac{\sin kx}{k}$$

- $f(x) = \frac{(x-\pi)^2}{4}$, $x \in [0, 2\pi]$



$$\mathcal{F}[f](x) = \frac{\pi^2}{12} + \sum_{\substack{k=-\infty \\ (k\neq 0)}}^{\infty} \frac{e^{ikx}}{2k^2} = \frac{\pi^2}{12} + \sum_{k=1}^{\infty} \frac{\cos kx}{k^2}$$

- in HW10: $f(x) = |x-\pi|$, $x \in [0, 2\pi]$



$$\mathcal{F}[f](x) = \frac{\pi}{2} + \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} \frac{2}{\pi k^2} e^{ikx}$$

note: the smoother a fct., the faster the Fourier coefficients decay