Jacobs University
Fall 2018

October 1, 2018

Linear Algebra

Homework 4

Due on October 8, 2018

Problem 1 [2 points]

Let V_1, V_2, W be subspaces of a vector space V. Either prove or give a counterexample to the following assertions.

(a) If $V_1 + W = V_2 + W$ then $V_1 = V_2$.

(b) If $V = V_1 \oplus W$ and $V = V_2 \oplus W$ then $V_1 = V_2$.

Problem 2 [6 points]

Let $V_1, \ldots, V_n \subset V$ be subspaces of the vector space V with $\sum_{i=1}^{n} V_i = V$. Prove that the following three statements are equivalent:

(a)

$$V = \bigoplus_{i=1}^{n} V_i \quad \text{(i.e., } V \text{ is actually a \textit{direct} sum of the } V_i\text{)},$$

(b)

$$V_j \cap \left(\sum_{\substack{i=1 \\ i \neq j}}^{n} V_i \right) = \{0\},$$

(c)

$$\sum_{i=1}^{n} \dim V_i = \dim V.$$

Problem 3 [4 points]

Let V be a vector space and let $p_1, \ldots, p_n \in \mathcal{L}(V)$ be projectors with $\sum_{i=1}^{n} p_i = \mathrm{id}$ and $p_i p_j = 0$ for all $i \neq j$. Prove that then

$$V = \bigoplus_{i=1}^{n} \operatorname{im}(p_i).$$

Problem 4 [4 points]

Let V be a vector space and let $p \in \mathcal{L}(V)$ be a projector. Prove that $V = \ker(p) \oplus \operatorname{im}(p)$. Show also that any projector can be represented in an appropriate basis by the matrix

$$A_p = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix},$$

where E_r is the $r \times r$ unit matrix with $r = \dim \operatorname{im}(p)$, and the 0's stand for $(n-r) \times (n-r)$ matrices with 0 entries ($n = \dim V$).

Problem 5 [4 points]

Let (V_1, V_2, V_3) be an ordered triple of pairwise distinct planes in $\mathbb{R}^3$. Prove that there exist two possible types of relative arrangements of such triples, characterized by $\dim V_1 \cap V_2 \cap V_3$. Which type should be regarded as the general one?