Jacobs University
Fall 2018

October 29, 2018

Linear Algebra

Homework 6

Due on November 5, 2018

Problem 1 [6 points]
Let $f \in \mathcal{L}(L)$ be diagonalizable with simple spectrum.

(a) Prove that any operator $g \in \mathcal{L}(L)$ such that $gf = fg$ can be represented in the form of a polynomial of f.

(b) Prove that the dimension of the space of such operators g equals the dimension of L.

(c) Are (a) and (b) still true if the spectrum of f is not simple?

Problem 2 [6 points]
Let $f, g \in \mathcal{L}(L)$ with $\dim(L) = n$, where L is a vector space over a field F with characteristic zero (meaning that no matter how often we add up the 1 of the field we will never get to the 0 of the field). Assume that $f^n = 0$, $\dim \ker(f) = 1$, and $gf - fg = f$. Prove that the eigenvalues of g have the form $a, a-1, a-2, \ldots, a-(n-1)$ for some $a \in F$.

Problem 3 [2 points]
Suppose $f, g \in \mathcal{L}(L)$ with $\dim(L) = n$, and L is a vector space over $\mathbb{C}$. Let all eigenvalues and eigenvectors of f and g coincide. Does this imply that $f = g$?

Problem 4 [3 points]
Let $J_r(\lambda)$ be the $r \times r$ Jordan block with λ on the diagonal. Find a general formula for $J_r(\lambda)^n$ for any $n \in \mathbb{N}$. Explicitly write down what the matrix $J_3(\lambda)^n$ is. Show that the characteristic polynomial and minimal polynomial of $J_r(\lambda)$ are the same. Also, find a counter example of some matrix where the characteristic polynomial is not equal to the minimal polynomial.

Problem 5 [3 points]
Find the eigenvalues and generalized eigenvectors of $f(x, y) = (-y, x)$, where $f \in \mathcal{L}(\mathbb{C}^2)$.

Finally: A Bonus Problem! [3 extra points]
Prove that for an arbitrary complex matrix one can introduce infinitesimal changes of the matrix elements that make the matrix diagonalizable.