Jacobs University
Fall 2018

November 5, 2018

Linear Algebra

Homework 7

Due on November 12, 2018

Problem 1 [**4 points**]
For the following two matrices

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix},$$

find the eigenvalues, their multiplicities, the corresponding eigenspaces or generalized eigenspaces and state whether they are diagonalizable or not. If a matrix is not diagonalizable, write down the Jordan normal form (only the matrix is enough).

Problem 2 [**3 points**]
Solve the linear system of equations

$$y'(t) = J_n(\lambda)y(t),$$

where $y(t) \in \mathbb{R}^n$, and $J_n(\lambda)$ is a Jordan block with $\lambda \in \mathbb{R}$ on the diagonal. *Hint: One could do this by making a guess for the solution and then prove it by induction.*

Problem 3 [**2 points**]
Consider the linear system of equations

$$y'(t) = Ay(t),$$

where $y(t) \in \mathbb{R}^n$ and A is any complex $n \times n$ matrix. Describe how to solve this equation by bringing A into Jordan normal form.

Problem 4 [**6 points**]
Let L be a finite dimensional complex vector space and $g \in \mathcal{L}(L)$. An operator $f \in \mathcal{L}(L)$ such that $f^2 = g$ is called a square root of g.

(a) Suppose g is nilpotent. Prove that then $1 + g$ has a square root. *Hint: Make a guess of what the square root should be from the Taylor series expansion for* $\sqrt{1+x}$.

(b) Now suppose that g is invertible. Prove that then g has a square root. *Hint: Start with the abstract Jordan decomposition and consider* $g|_{G(\lambda_i)}$.

Problem 5 [5 points]

Let L be a real or complex vector space.

(a) Recall or look up the definition of a norm $||\cdot||$ on L. Let us next consider any norm for linear operators $f : L \to L$. Let $P(t) = \sum_{i=0}^{\infty} c_i t^i$ have non-zero radius of convergence. We set $P(f) = \sum_{i=0}^{\infty} c_i f^i$ if the series $\sum_{i=0}^{\infty} c_i ||f^i||$ converges (in $\mathbb{R}$).

(b) Let $f \in \mathcal{L}(L)$ be such that $||f|| < 1$. Compute $\sum_{i=0}^{\infty} f^i$ and thus prove that $1 - f$ is invertible.

(c) Let $f, g \in \mathcal{L}(L)$. We define $e^f = \sum_{i=0}^{\infty} \frac{f^i}{i!}$. Prove that when $fg = gf$ (i.e., f and g commute), then $e^{f+g} = e^f e^g$. *Hint: Cauchy product.*