Jacobs University
Fall 2018

November 12, 2018

Linear Algebra

Homework 8

Due on November 19, 2018

Problem 1 [5 points]
Let L and M be finite dimensional linear spaces over the field F and let $g : L \times M \to F$ be a bilinear mapping. We shall call the set

$$L_0 = \{\ell \in L : g(\ell, m) = 0 \text{ for all } m \in M\}$$

the left kernel of g and the set

$$M_0 = \{m \in M : g(\ell, m) = 0 \text{ for all } \ell \in L\}$$

the right kernel of g. Prove the following assertions:

(a) $\dim L/L_0 = \dim M/M_0$.

(b) g induces the bilinear mapping $g' : L/L_0 \times M/M_0 \to F, g'(\ell + L_0, m + M_0) = g(\ell, m)$, for which the left and right kernels are zero.

Problem 2 [3 points]
Prove that any bilinear inner product $g : L \times L \to F$ (over the field F with characteristic $\neq 2$) can be uniquely decomposed into a sum of symmetric and antisymmetric inner products.

Problem 3 [8 points]
Let $g : L \times L \to F$ be a bilinear inner product such that the property of orthogonality of a pair of vectors is symmetric, i.e., from $g(\ell_1, \ell_2) = 0$ it follows that $g(\ell_2, \ell_1) = 0$. Prove that then g is either symmetric or antisymmetric.

You can proceed in the following way:

(a) Let $\ell, m, n \in L$. Prove that $g\big(\ell, g(\ell, n)m - g(\ell, m)n\big) = 0$. Using the symmetry of orthogonality, deduce that $g(\ell, n)g(m, \ell) = g(n, \ell)g(\ell, m)$.

(b) Set $n = \ell$ and deduce that if $g(\ell, m) \neq g(m, \ell)$ then $g(\ell, \ell) = 0$.

(c) Show that $g(n, n) = 0$ for any vector $n \in L$ if g is non-symmetric. To this end, choose ℓ, m with $g(\ell, m) \neq g(m, \ell)$ and study separately the cases $g(\ell, n) \neq g(n, \ell)$ and $g(\ell, n) = g(n, \ell)$.

(d) Show that if $g(n,n) = 0$ for all $n \in L$ then g is antisymmetric.

Problem 4 [4 points]

Let (L, g) be an n-dimensional linear space with a non-degenerate inner product. Prove that the set of vectors $\{e_1, \ldots, e_n\}$ in L is linearly independent if and only if the matrix $(g(e_i, e_j))_{i,j}$ is non-singular.

Bonus Problem [3 extra points]

Give the classification of one-dimensional orthogonal spaces over a finite field F with characteristic $\neq 2$ by showing that $F^*/(F^*)^2$ is the cyclical group of order 2. (Hint: Show that the kernel of the homomorphism $F^* \to F^* : x \mapsto x^2$ is of order 2, using the fact that the number of roots of any polynomial over a field does not exceed the degree of the polynomial.)