Linear Algebra

Organization:

- syllabus, website
- weekly homework sheets (Wed, starting Sep. 12)
  - ↳ rules as usual (syllabus)
  - ↳ due on Wed. before class (mailbox)
  - ↳ 2 worst sheets disregarded for grading
- weekly tutorials (doodle poll)
- TA: Khadeeja Afzal
- my and her office hours: TBA
- 2 exams: · midterm (Wed, Oct. 24, after Reading Days)
  - · final (in final exam period, Dec.)

- grades: 20% HW
  - 30% midterm
  - 50% final

If final grade > midterm grade $\Rightarrow$ midterm grade := final grade

- Topics: • vector spaces and linear operators
  (linear spaces and linear mappings)

  ↳ subspaces, basis, dimension, dual space, quotient spaces

  ↳ fundamental spaces, eigenvalues, eigenvectors, characteristic polynomial, Jordan decomposition, (de)complexification

  • geometry: inner products or bilinear/sesquilinear form

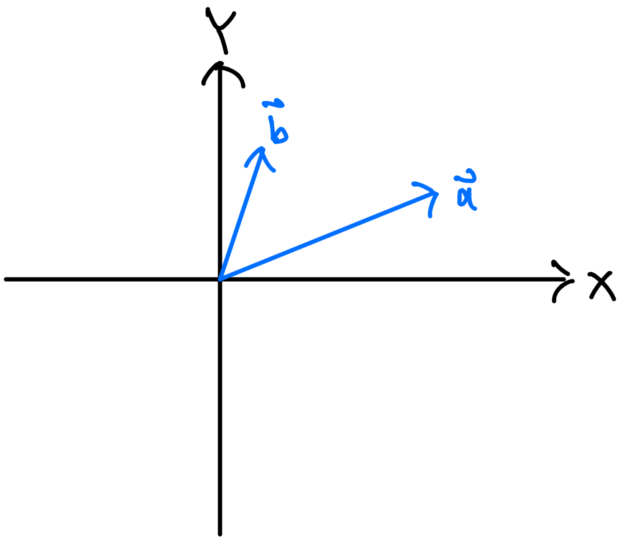
  ↳ Euclidean, Hermitian, Symplectic

- books: Kostrikin, Manin ; Axler

I. Vector Spaces and Linear Operators

I.1 Vector Spaces

motivation/origin: vectors in $\mathbb{R}^2, \mathbb{R}^3$ (forces, trajectories, EM fields, ...)



→ addition (commutative, associative, zero vector, inverse)

→ scaling or scalar multiplication (identity, distributive)

now: make both vectors and scalars abstract

Definition: A field $\mathbb{F}$ is a set with addition and multiplication, both of which are associative, commutative, have identities and inverses, and are distributive.

Examples:
- $\mathbb{R}$ (real numbers)
- $\mathbb{C}$ (complex numbers)
- $\mathbb{Q}$ (rational numbers)

$\Big\}$ most often used

Def.: A vector space V over a field $\mathbb{F}$ is a set with the two operations addition (+) and multiplication ($\cdot$) by an elements in $\mathbb{F}$ (scalars) that satisfies the following axioms:

(- for addition:) 1) Associativity: $(v_1+v_2)+v_3 = v_1+(v_2+v_3)$

$\forall v_1, v_2, v_3 \in V$

2) Existence of an identity called 0 (or "zerovector" or "neutral element"): $v_1 + 0 = v_1 \quad \forall v_1 \in V$

3) Existence of an inverse: for any $v_1 \in V$ $\exists$ inverse $-v_1$, such that $v_1 + (-v_1) = 0$

4) Commutativity: $v_1+v_2 = v_2+v_1 \quad \forall v_1, v_2 \in V$

(note: 1) - 4) are the properties of an abelian group)

(- for scalar multiplication:)

5) Associativity: $(\alpha\beta)\cdot v_1 = \alpha\cdot(\beta v_1)$

$\forall \alpha, \beta \in \mathbb{F}, v_1 \in V$

6) Distributivity for scalars: $\alpha\cdot(v_1+v_2) = \alpha\cdot v_1 + \alpha\cdot v_2$

$\forall \alpha \in \mathbb{F}, v_1, v_2 \in V$

7) Distributivity for vectors: $(\alpha+\beta)\cdot v_1 = \alpha\cdot v_1 + \beta\cdot v_1$

$\forall \alpha, \beta \in \mathbb{F}, v_1 \in V$

8) Multiplicative identity $1 \in \mathbb{F}$: $1 \cdot v_1 = v_1$, $\forall v_1 \in V$

Remarks:

- 0 identity is unique

Proof: suppose $\exists\, 0_1$ and 0_2. Then $0_1 = 0_1 + 0_2 = 0_2 + 0_1 = 0_2$.

- $0 \cdot \alpha = 0 \;\; \forall \alpha \in \mathbb{F}$, $0 \cdot v_1 = 0 \;\; \forall v_1 \in V$

Proof: $0 \cdot v_1 + 0 \cdot v_1 = (0+0) \cdot v_1 = 0 \cdot v_1$.

- inverse is unique, $-v_1 = (-1) \cdot v_1$

Proof: $v_1 + (-1) v_1 = 1 \cdot v_1 + (-1) \cdot v_1 = (1 + (-1)) \cdot v_1 = 0 \cdot v_1 = 0$.

- $\alpha \cdot v_1 = 0 \Rightarrow \alpha = 0$ or $v_1 = 0$

Proof: say, $\alpha \neq 0$. Then $0 = \alpha^{-1}(\alpha \cdot v_1) = (\alpha^{-1}\alpha) \cdot v_1 = 1 \cdot v_1 = v_1$.

Examples:

- $\mathbb{R}^n$, $a \in \mathbb{R}^n$ then $a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$, $a_i \in \mathbb{R}$

addition: $a + b = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix}$

scalar mult.: $\alpha \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \alpha \cdot a_1 \\ \vdots \\ \alpha \cdot a_n \end{pmatrix}$

(note: at this point we don't have lengths, angles, areas, volumes etc.; comes later with inner products)

- $\mathbb{C}^n$: same with $a_i \in \mathbb{C}$
- $Mat_{n\times m}(\mathbb{R})$ = space of $n\times m$ real matrices
- space of functions in $\mathbb{R}$; $f: \mathbb{R}\to\mathbb{R}$, $g: \mathbb{R}\to\mathbb{R}$, $\alpha, x \in \mathbb{R}$

  add.: $(f+g)(x) = f(x)+g(x)$

  scalar mult.: $(\alpha f)(x) = \alpha \cdot f(x)$
- space of fct. $X\to\mathbb{R}$ for any set X
- space of all linear fct.s $V\to F$
- polynomials, e.g., $Pol_{\leq n}(\mathbb{R}\to\mathbb{R})$, $Pol_{\leq n}(\mathbb{C}\to\mathbb{C})$, $Pol(\mathbb{R})$

  $\hookrightarrow x \mapsto a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$

  for $a_0, \dots, a_n \in \mathbb{R}$

I.2 Subspace

Ex.: • lines or planes in $\mathbb{R}^3$

• $Pol_{\leq n}(\mathbb{R}) \subset Pol(\mathbb{R}) \subset C^\infty \subset \dots \subset C^2 \subset C^1 \subset D \subset C \subset \underbrace{F(\mathbb{R})}_{\text{all fct.s } \mathbb{R}\to\mathbb{R}}$

D: differentiable fct.s

Def.: A subset $W \subset V$ is called a subspace if W is a vector space w.r.t. the induced operations (i.e., the same addition and scalar mult. as in V).