Session 2, Sep. 10, 2018

Subspaces:

motivated by lines, planes (through the origin) in $\mathbb{R}^3$ and results from Analysis (e.g. sums of cont. fct.s are cont.)

<u>Def.:</u> A subset $W \subset V$ is called a subspace if W is a vector space w.r.t. the induced operations (i.e., the same addition and scalar mult. as in V).

non-empty subset $W \subset V$ is a subspace

$\iff$ W is closed w.r.t. the induced operations

<u>Proof:</u> "$\Rightarrow$" clear

"$\Leftarrow$" • $w_1 \in W \Rightarrow \underset{\substack{\uparrow \\ \text{scalar}}}{0} \cdot w_1 = 0 \in W$

• $w_1 \in W \Rightarrow (-1) w_1 = -w_1 \in W$

• rest follows bc. it holds on V

<u>note:</u> • neutral element 0_V of V equals neutral element 0_W of W

<u>Proof:</u> $w \in W \Rightarrow w + 0_W = w = w + 0_V$

if $-w_V$ the inverse of w in V, then we get $w + 0_W - w_V = w + 0_V - w_V$

$\iff 0_V + 0_W = 0_V + 0_V$

$\iff 0_W = 0_V$

- let $w \in W$, then inv. $-w_w$ in W equals inv. $-w_v$ in V

Proof: $w - w_w = 0 = w - w_v$

$$\Longleftrightarrow w - w_w - w_v = w - w_v - w_v$$

$$-w_w = -w_v$$

Ex.:
- $\{(0, x_2, x_3) \in \mathbb{R}^3\}$ is a subspace
- $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 = x_1 + 3x_3 - 5\}$ is not a subspace
- space of cont. fct.s is a subspace of space of all fct.s

I.3 Span, Basis, Dimension

motivation:

- two vectors $v_1, v_2 \in \mathbb{R}^3$ with $v_1 \neq c \cdot v_2$ for $c \in \mathbb{R}$ span a (unique) plane through 0

  ↳ then all vectors in plane can be written as $w = c_1 v_1 + c_2 v_2$ $(c_1, c_2 \in \mathbb{R})$

- all planes through 0 can be written as $c_1 x + c_2 y + c_3 z = 0$ for non-zero c_1, c_2, c_3

Def.: Let V be a vectorspace over some field F.
Then $c_1 v_1 + \dots + c_k v_k = \sum_{i=1}^{k} c_i v_i$ for $c_i \in F, v_i \in V, i = 1, \dots, k$ is called a linear combination.

Def.: For any subset $S \subset V$, we def.

$$\mathrm{span}(S) = \left\{ \text{all linear combinations } \sum_{i=1}^{k} c_i s_i \text{ with } c_i \in F, s_i \in S, \; i=1,\dots,k, \; k \in \mathbb{N} \right\}$$

note: • for any subset $S \subset V$, $\mathrm{span}(S)$ is a subspace (HW)

• $\mathrm{span}(S)$ is the smallest subspace, i.e., $\mathrm{span}(S) = \bigcap_{\substack{W \subset V \text{ subspace} \\ \text{and } S \subset W}} W$ (HW)

• if $W \subset V$ is a subspace, then $\mathrm{span}(W) = W$.

Def.: A subset $S \subset V$, s.t. $\mathrm{span}(S) = V$ is called a generating set (or, a spanning set).

Def.: A minimal generating set is called a basis of V (minimal means no element can be removed).

Def.: A subset $S \subset V$ is called linearly independent if $\sum_{i=1}^{k} c_i s_i = 0$ for $c_i \in F$, $s_i \in S$ always implies $c_i = 0 \; \forall i = 1,\dots,k$

Otherwise it's called linearly dependent.

__Theorem:__ Let V be a vector space over a field F, and $E \subset V$ a subset. Then the following are equivalent:

1) E is a basis of V

2) E is a maximal linearly independent set

3) every $v \in V$ can uniquely be written as $v = \sum_{i=1}^{k} c_i e_i$, for $c_i \in F$, $e_i \in E$, $i = 1, \dots, k$, $k \in \mathbb{N}$

__Proof:__

"1) => 2)": We suppose E is a basis of V.

Assume E is linearly dependent. Then $\exists\, c_1, \dots, c_k \in F$, $e_1, \dots, e_k \in E$,

s.t. $\sum_{i=1}^{k} c_i e_i = 0$ and $c_1 \neq 0$.

$\Rightarrow e_1 = -\sum_{i=2}^{k} \frac{c_i}{c_1} e_i \Rightarrow E \setminus \{e_1\}$ is a generating set

$\Rightarrow$ contradiction to E is a basis (a __minimal__ generating set)

$\Rightarrow E$ is linearly independent

Maximal? Choose $v \notin E$, can $E \cup \{v\}$ be linearly indep.?

Since E is generating, $v = \sum_{i=1}^{k} c_i e_i$ for some $c_i \in F$, $e_i \in E$, $i = 1, \dots, k$, $k \in \mathbb{N}$

$\Rightarrow \quad 1 \cdot v + \sum_{i=1}^{k} (-c_i) e_i = 0 \Rightarrow E \cup \{v\}$ is lin. dep.

$\Rightarrow$ E is maximal lin. indep. set

"2) => 3)": We suppose E is max. lin. indep. set

choose $v \in V$, if $v \in E$ we are done, so suppose $v \notin E$.

Then $E \cup \{v\}$ is lin. dep., i.e., $\exists\, c, c_1, c_2, \ldots, c_k \in F$ and $e_1, \ldots, e_k \in E$, s.t. $c \cdot v + \sum_{i=1}^{k} c_i e_i = 0$ and not all $c_i = 0$.

$\Rightarrow v = -\sum_{i=1}^{k} \frac{c_i}{c} e_i$ ($c \neq 0$, otherwise E wouldn't be lin. indep.)

Uniqueness?

suppose $\exists\, e_1, \ldots, e_k \in E$, $c_1, \ldots, c_k \in F$, $d_1, \ldots, d_k \in F$, such that

$$v = \sum_{i=1}^{k} c_i e_i = \sum_{i=1}^{k} d_i e_i$$

$$\Rightarrow \sum_{i=1}^{k} (c_i - d_i) e_i = 0$$

$\Rightarrow c_i = d_i \;\; \forall i = 1, \ldots, k$, since E is linearly indep.

"3) => 1)": We suppose any $v \in V$ can be uniquely written as $\sum_{i=1}^{k} c_i e_i$.

Then clearly E is a generating subset of V. Is it also minimal?

Assume E not minimal.

Then $\exists\, e \in E$ s.t. $E \setminus \{e\}$ is a generating subset.

$\Rightarrow \exists\, e_1, \ldots, e_k \in E$, $c_1, \ldots, c_k \in F$, s.t. $e = \sum_{i=1}^{k} c_i e_i$.

Thus, e can be written in two ways

$\Rightarrow$ contradiction to unique representation $\Rightarrow E$ minimal $\quad \square$