Session 3
Sep. 12, 2018

Last time: subspaces, lin. combinations, lin. (in)dependence, basis

E basis (minimal generating set) $\iff$ E max. lin. indep. $\iff$ every vector can be written as unique lin. comb.

<u>Ex.:</u> • $V = \mathbb{F}^n$ (e.g., $\mathbb{R}^n$, $\mathbb{C}^n$) with componentwise addition and scalar multiplication
↳ as vector space over $\mathbb{F}$

one basis is $E = \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$, the canonical or standard basis of $\mathbb{F}^n$

$$V \ni v = \sum_{i=1}^{n} c_i e_i$$

• $V = \mathrm{Mat}_{m\times n}(\mathbb{F})$, one basis is $E = \{ e_{ij}, i = 1, \dots, m, j = 1, \dots, n \}$

$$e_{ij} = \begin{array}{c} \begin{array}{ccc} & i & \end{array} \\ \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix} \end{array} j \quad , \quad V \ni M = \sum_{i=1}^{m} \sum_{j=1}^{n} \underbrace{m_{ij}}_{\in \mathbb{F}} \underbrace{e_{ij}}_{\in V}$$

• $V = \mathbb{F}(S)$, the space of fct.s from $S \to \mathbb{F}$, say $|S| = n$ (S has n elements)

one basis is $E = \{ \delta_s, s \in S \}$, where $\delta_s(x) = \begin{cases} 1 & \text{for } x = s \\ 0, & \text{for } x \neq s \end{cases}$

(if $S = \{1, \dots, n\}$, then $\delta_s(x) = \delta_{sx}$, the Kronecker delta)

$V \ni f = \sum_{s \in S} c_s \delta_s$, so $f(x) = c_x$

- $V = \mathrm{Pol}(\mathbb{F}) = \mathbb{F}[x]$, space of polynomials with coefficients in $\mathbb{F}$, let's say $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

a basis is $E = \{1, x, x^2, x^3, \dots\}$

clearly this is generating. Is it also minimal?

Is $E \setminus \{x^k\}$ still generating?

If yes, $\exists\, c_{i_0}, c_{i_1}, \dots, c_{i_n} \in \mathbb{F}$, with $\{i_0, \dots, i_n\} \subset \mathbb{N}$, $i_j \neq k \;\forall j = 0,\dots,n$

s.t. $x^k = \sum_{j=0}^{n} c_{i_j} x^{i_j} \Rightarrow f(x) = \sum_{j=0}^{n} c_{i_j} x^{i_j} - x^k = 0 \;\forall x \in \mathbb{F}$

$\Rightarrow$ factorize or polynomial division: $f(x) = (x - x_0) g(x)$ for any $x_0 \in \mathbb{F}$

repeat for all $x_0 \in \mathbb{F} \Rightarrow$ contradiction to finite degree.

<u>Def.:</u> V is called finite dimensional if it has a finite basis. If not, it is called infinite dimensional.

To def. dimension in a meaningful way, we need:

<u>Thm.:</u> If V is finite dimensional, then the number of elements in a basis, does not depend on the basis.

Def.: If V has a basis with n elements, then n is called the dimension of V, i.e., $\dim V = n$ (or $\dim_F V = n$).

Proof of thm.: Let $\{e_1, \dots, e_n\}$ and $\{e_1', \dots, e_m'\}$ be bases with $m > n$.

$$
\begin{aligned}
e_1' &= a_{11} e_1 + a_{12} e_2 + \dots + a_{1n} e_n \\
&\vdots \\
e_m' &= a_{m1} e_1 + a_{m2} e_2 + \dots + a_{mn} e_n
\end{aligned}
$$

$$
\begin{aligned}
e_1 &= b_{11} e_1' + \dots + b_{1m} e_m' \\
&\vdots \\
e_n &= b_{n1} e_1' + \dots + b_{nm} e_m'
\end{aligned}
$$

or with $e = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$, $e' = \begin{pmatrix} e_1' \\ \vdots \\ e_m' \end{pmatrix}$

$$
A = \underbrace{\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}}_{m \times n}
\qquad
B = \underbrace{\begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nm} \end{pmatrix}}_{n \times m}
$$

$$
e' = Ae, \quad e = Be' \quad \begin{array}{l} \nearrow\ e = Be' = BAe \\ \Longrightarrow e' = Ae = ABe' \end{array}
$$

$$\Rightarrow BA = E_n = \underbrace{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_{n \times n} \quad , AB = E_m = \underbrace{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_{m \times m}$$

now we could argue with rank of matrices

rank $A \leq n$, rank $B \leq n$ $\Rightarrow$ rank $AB \leq n$, rank $BA \leq n$

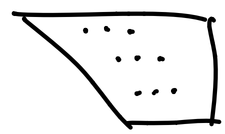
but $AB = E_m$, which has rank $m > n$.

alternatively: argue with uniqueness of 0:

$$0 = \sum_{i=1}^{m} c_i e'_i = \sum_{i=1}^{m} c_i \sum_{j=1}^{n} a_{ij} e_j = \sum_{j=1}^{n} \underbrace{\left(\sum_{i=1}^{m} c_i a_{ij} \right)}_{= 0 \text{ (uniqueness of } 0 \text{ in } e)} e_j$$

$\sum_{i=1}^{m} c_i a_{ij} = 0$ for $j = 1, \dots, n$ is a system of n eq.s for m unknowns $c_1, \dots, c_m$, with $m > n$. But this always has a non-zero solution. $\square$

can be seen, e.g., by performing Gaussian elimination



Ex.: $\cdot \dim \mathbb{F}^n = n$

$\cdot \dim \mathbb{F}(S) = |S|$, if S is finite

$\cdot \dim_{\mathbb{C}} \mathbb{C} = 1$, $\dim_{\mathbb{R}} \mathbb{C} = 2$ (dim can depend on field) note: $\mathbb{R}$ as vector space over $\mathbb{Q}$ is infinite dim. (HW)

$\mathbb{R}$ over $\mathbb{Q}$ actually has a basis, called Hamel basis, which is uncountable, and not explicitly given.

↳ infinite dim. vector spaces are more interesting with topologies (e.g., Banach space, Hilbert space)

<u>But:</u>

<u>Thm.:</u> Every vector space has a basis.

Proof requires Zorn's Lemma. Next time...