Session 4
Sep. 17, 2018

From the axioms of set theory (Zermelo-Fraenkel) Zorn's Lemma can neither be proven nor disproven.

(It is equivalent to axiom of choice and to well-ordering principle.)

<u>Zorn's Lemma:</u> Let S be a partially ordered set, s.t. every linearly ordered subset has an upper bound. Then S has a maximal element.

Explanation:

- <u>S partially ordered means:</u>

S has a relation $\leq$ that satisfies $\forall s_1, s_2, s_3 \in S$:

  - $s_1 \leq s_1$ (reflexive)
  - $s_1 \leq s_2$ and $s_2 \leq s_1 \Rightarrow s_1 = s_2$ (antisymmetric)
  - $s_1 \leq s_2$ and $s_2 \leq s_3 \Rightarrow s_1 \leq s_3$ (transitive)

(not: not necessarily $s_1 \leq s_2$ or $s_2 \leq s_1$; s_1 and s_2 can be incomparable)

- <u>X linearly ordered means:</u>

$x_1, x_2 \in X$: $x_1 \leq x_2$ or $x_2 \leq x_1$ (two elements are always comparable)

- <u>$X \subset S$ has upper bound means:</u>

$\exists y \in S$ s.t. $x \leq y \; \forall x \in X$

- S has max. element:

$s \in S$ is a max. element if $\forall s' \in S$ for which $s \leq s' \Rightarrow s' = s$

Proof of thm. (every vector space has a basis):

Let V be the vector space.

Let $S := \{\text{subsets } T \subset V, T \text{ linearly independent}\}$

partial order on S by inclusion ("$\leq$" is subset inclusion "$\subseteq$": $S_1 \subseteq S_1$;
$S_1 \subseteq S_2$ and $S_2 \subseteq S_1 \Rightarrow S_1 = S_2$;
$S_1 \subseteq S_2$ and $S_2 \subseteq S_3 \Rightarrow S_1 \subseteq S_3$)

Let X be a linearly ordered subset of S. Does it have an upper bound?

Yes, $X_u = \bigcup_{Y \in X} Y$ is clearly an upper bound ($x \in X_u \ \forall x \in X$).

Choose any finite set of vectors $\{x_1, \dots, x_n\}$ from X_u, then these are contained in one of the $Y \in X \Rightarrow X_u$ still linearly independent.
$\Rightarrow X_u \in S$

Use Zorn's Lemma: S has a maximal element, i.e., there is a lin. indep. subset $E \in S$ that is maximal, i.e., no element can be added.

$\Rightarrow E$ is a basis. $\square$

I.4 Linear Maps

$m: S_1 \to S_2$ is called a map or function (sometimes fct. only if $S_2 = \mathbb{Q}, \mathbb{R}, \mathbb{C}$)

If V, W are vector spaces over a field F, $f: V \to W$ is called linear if

$$f(c_1 v_1 + c_2 v_2) = c_1 f(v_1) + c_2 f(v_2) \quad \forall c_1, c_2 \in F, v_1, v_2 \in V.$$

("linear map" or "linear operator" (Axler: "operator" only if $V=W$))

$\mathcal{L}(V, W) = \{$all linear maps from V to $W\}$

$\mathcal{L}(V) = \{$all lin. maps from V to $V\}$

Ex.: • zero map $0 \in \mathcal{L}(V, W)$: $\underbrace{0}_{\in \mathcal{L}(V,W)} \overbrace{v}^{\in V} = 0(v) = \underbrace{0}_{\in W} \quad \forall v \in V$

• F^∞ = set of all sequences in F

e.g., $f(x_1, x_2, \ldots) := (x_2, x_3, \ldots)$ "left" or "backward" shift is linear

• $V = C^1(\mathbb{R})$ = cont. differentiable fct.s

derivative D is linear $((f+g)' = f' + g', \ (\lambda f)' = \lambda f')$

note: • addition: $f_1, f_2 \in \mathcal{L}(V, W)$, $v \in V$, then $(f_1 + f_2)(v) := f_1(v) + f_2(v)$

• mult.: $c \in F$: $(cf_1)(v) = c f_1(v)$

Lemma: $\mathcal{L}(V,W)$ is a vector space.

Proof: HW

Lemma: Let $\{v_1,\dots,v_n\}$ be a basis in V (in particular, $\dim V = n < \infty$) and $\{w_1,\dots,w_n\} \subset W$. Then there exists a unique linear map $f: V \to W$ with $f(v_i) = w_i \ \forall i = 1,\dots,n$.

Proof: • uniqueness: $f, f' \in \mathcal{L}(V,W)$ with $f(v_i) = w_i$, $f'(v_i) = w_i$.

$g := f - f'$ is still linear

$$g\left(\sum_{i=1}^n c_i v_i\right) = \sum_{i=1}^n c_i\, g(v_i) = \sum_{i=1}^n c_i\big(\underbrace{f(v_i)}_{=w_i} - \underbrace{f'(v_i)}_{=w_i}\big) = 0$$

$\Rightarrow g = 0$ (zero map) $\Rightarrow f = f'$

• existence: Def. f by $f\left(\sum_{i=1}^n c_i v_i\right) := \sum_{i=1}^n c_i w_i$ (clearly linear)

$\square$

If V vector space over $\mathbb{F}$, then elements of $\mathcal{L}(V,\mathbb{F})$ are called linear fct.s of V or functionals of V.

$\mathcal{L}(V,\mathbb{F}) =: V^*$ is called the dual space of V

Ex.: • $\mathcal{L}(\mathbb{F}^n, \mathbb{F}) = (\mathbb{F}^n)^*$

usual notation: $\mathbb{F}^n \ni \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $(\mathbb{F}^n)^* \ni (x_1^*, \dots, x_n^*)$

e.g., $(x_1^*, \dots, x_n^*) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^n x_i^* x_i$ (matrix multiplication)

Def.: Let $\{v_1, \dots, v_n\}$ be a basis of V ($\dim V = n < \infty$). Then $\{v_1^*, \dots, v_n^*\} \subset V^*$, with $v_i^*\left(\sum_{j=1}^n c_j v_j\right) = c_i$ $\left(v_i^*(v_j) = \delta_{ij}\right)$ is called the dual basis.

Lemma: The dual basis is indeed a basis.

Proof: $\{v_1^*, \dots, v_n^*\}$ generates V^*:

Pick $f \in V^*$, call $f(v_i) = f_i \in \mathbb{F}$, $f = \sum_{i=1}^n f_i v_i^*$.

$$f\left(\sum_{i=1}^n c_i v_i\right) = \sum_{i=1}^n c_i f(v_i) = \sum_{i=1}^n c_i f_i = \left(\sum_{j=1}^n f_j v_j^*\right)\left(\sum_{i=1}^n c_i v_i\right)$$

Also from $\sum_{i=1}^n c_i v_i^* = 0$ it follows that

$$\left(\sum_{i=1}^n c_i v_i^*\right)(v_j) = c_j = 0 \quad \forall j = 1, \dots, n$$

$\Rightarrow$ all c_j's $= 0$ $\Rightarrow$ $\{v_1^*, \dots, v_n^*\}$ lin. indep. $\square$

But: In ∞-dim. vector space, $\{v_1^*, v_2^*, \dots\}$ is still lin. indep., but not generating anymore (not a basis) $\longrightarrow$ HW