Session 6
Sep. 24, 2018

Dual map:

(sometimes $f': W' \to V'$)

Let $f: V \to W$ be linear. Then dual map $f^*: W^* \to V^*$ is def. by

$$\underbrace{f^*(w^*)(v)}_{\in \mathbb{F}} = \underbrace{w^*(f(v))}_{\in \mathbb{F}} \quad \forall v \in V.$$

(later: inner products e.g., on Hilbert spaces $\langle f^* w, v \rangle = \langle w, f v \rangle$)

Ex.: consider derivative $D: \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R}),\ p \mapsto p'$

dual map $D^*(w^*)(p) = w^*(Dp) = w^*(p')$

e.g., $w^*(p) = \int_0^1 p(x)\,dx \Rightarrow D^*(w^*)(p) = w^*(p') = \int_0^1 p'(x)\,dx = p(1) - p(0)$

Thm.: Dual map is linear and unique.

Proof: uniqueness: Let f_1, f_2 be dual maps to f

$$\Rightarrow f_1^*(w^*)(v) = w^*(f(v)) = f_2^*(w^*)(v)$$

$$\Rightarrow (f_1^* - f_2^*)(w^*)(v) = 0 \quad \forall v \in V$$

$$\Rightarrow (f_1^* - f_2^*)(w^*) = 0 \quad \Rightarrow f_1^* = f_2^*$$

f^* indeed maps to V^*: fix $w^* \in W^* \Rightarrow w^*(f(v))$ as fct. of v

linear, i.e., $\in V^*$, since $w^*(f(c_1 v_1 + c_2 v_2)) = c_1 w^* f(v_1) + c_2 w^* f(v_2)$
(linearity of f, w^*)

f^* is linear, since $f^*(c_1 w_1^* + c_2 w_2^*)(v) \underset{\uparrow}{=} c_1 f^*(w_1^*) + c_2 f^*(w_2^*)$.

by def. + lin. of w_1^*, w_2^*

$\square$

Kernel and image of linear maps:

<u>Def.:</u> Let $f: V \to W$ be linear. Then

- kernel or null space is $\ker f = \{v \in V : f(v) = 0\} \subset V$
- image or range is $\operatorname{im} f = \{w \in W : \exists v \in V \text{ with } f(v) = w\} \subset W$

note: $\ker f$ and $\operatorname{im} f$ are actually subspaces (HW)

<u>Ex.:</u>
- differentiation $D: \mathrm{Pol}(\mathbb{R}) \to \mathrm{Pol}(\mathbb{R})$

  $\ker D = \{g(x) = c \cdot 1 \quad \forall c \in \mathbb{F}\}$

  $\operatorname{im} D = \mathrm{Pol}(\mathbb{R})$

- backward shift $B: \mathbb{F}^\infty \to \mathbb{F}^\infty$

  $\ker B = \{(c, 0, 0, \dots), c \in \mathbb{F}\}$

  $\operatorname{im} B = \mathbb{F}^\infty$

<u>Lemma:</u> Let $f \in \mathcal{L}(V, W)$. Then f injective $\Leftrightarrow \ker f = \{0\}$.

<u>Proof:</u> "$\Rightarrow$" take $v \in \ker f$

$\Rightarrow f(v) \underset{\uparrow}{=} 0 \underset{\nwarrow}{=} f(0) \Rightarrow v = 0$, since f injective

$v \in \ker f$ $\quad$ f linear

"$\Leftarrow$" suppose $v_1, v_2 \in V$ with $f(v_1) = f(v_2)$

$$\Rightarrow f(v_1) - f(v_2) \overset{f \text{ lin.}}{=} f(v_1 - v_2) = 0 \Rightarrow v_1 - v_2 \in \ker f$$

$$\Rightarrow v_1 = v_2, \text{ since } \ker f = \{0\} \qquad \square$$

Thm. ("Fundamental Thm. of Linear Maps", "Rank-Nullity Thm."):

Let $f \in \mathcal{L}(V,W)$ with V finite dimensional. Then

$$\dim V = \underbrace{\dim(\operatorname{im} f)}_{\operatorname{rank}(f)} + \underbrace{\dim(\ker f)}_{\operatorname{nullity}(f)}.$$

Proof: note: $\dim(\ker f) < \infty$, since $\dim V < \infty$ and $\ker f$ subspace of V.

choose basis $\{v_1, \dots, v_m\}$ of $\ker f$, extend it to a basis

$\{v_1, \dots, v_m, v_{m+1}, \dots, v_n\}$ of V, where $n \geq m$

claim: $\{f(v_{m+1}), \dots, f(v_n)\}$ is a basis of $\operatorname{im} f$, this would prove thm.

- Does $\{f(v_{m+1}), \dots, f(v_n)\}$ generate $\operatorname{im} f$?

$$f\left(\sum_{i=1}^{n} c_i v_i\right) = \sum_{i=m+1}^{n} c_i f(v_i) \qquad \text{Yes.}$$

- Is $\{f(v_{m+1}), \dots, f(v_n)\}$ linear independent?

Let $\sum_{i=m+1}^{n} c_i f(v_i) = 0$

$$\Rightarrow f\left(\sum_{i=m+1}^{n} c_i v_i\right) = 0 \Rightarrow \sum_{i=m+1}^{n} c_i v_i \in \ker f$$

$\Rightarrow$ write it as $\sum_{i=m+1}^{n} c_i v_i = \sum_{i=1}^{m} c_i v_i$

$\Rightarrow$ all $c_i = 0$, since $\{v_1, \dots, v_n\}$ is basis of V $\square$

Corollary: let $f \in \mathcal{L}(V, W)$, $\dim V < \infty$. Then

$$f \text{ injective} \Leftrightarrow \dim V = \dim(\operatorname{im} f)$$

Proof: clear, since f inj. $\Leftrightarrow \ker f = \{0\} \Rightarrow \dim(\ker f) = 0$.

Corollary: let $f \in \mathcal{L}(V, W)$, $\dim V = \dim W < \infty$. Then

$$f \text{ isomorphism} \Leftrightarrow \ker f = \{0\} \Leftrightarrow \operatorname{im} f = W$$

Proof: clear $\ker f = \{0\} \Leftrightarrow \dim \operatorname{im} f = \dim V = \dim W \Leftrightarrow \operatorname{im} f = W$
(im f subspace)

$\Leftrightarrow f$ bijective.

Ex.: • recall $\varepsilon : V \to V^{**}$ with $\varepsilon(v)(f) = f(v)$ $\forall f \in V^*$

given $v \neq 0$, there is always $f \in V^*$, s.t. $f(v) \neq 0$

$\Rightarrow \ker \varepsilon = \{0\} \Rightarrow \varepsilon$ is an isomorphism

• let $f \in \mathcal{L}(F^n, F^m)$, consider $f(x_1, \dots, x_n) = 0$

(homogeneous system of linear eq.s)

let $n > m$ (more variables than eq.s)

Are there more solution than $(x_1, \ldots, x_n) = (0, \ldots, 0)$?

I.e., is $\ker f = \{0\}$ or not?

from thm.: $\dim \ker f = \underbrace{\dim V}_{n} - \underbrace{\dim \operatorname{im} f}_{\leq \dim W = m}$

$$\geq n - m > 0$$

$\Rightarrow$ there are always non-zero solutions