I.5 Matrices

Session 7
Sep. 26, 2018

$f \in \mathcal{L}(V, W)$, $\dim V = n < \infty$, $\dim W = m < \infty$

choose basis $B_v = \{v_1, \dots, v_n\}$ of V and $B_w = \{w_1, \dots, w_m\}$ of W

f det. by action on v_k, write it as lin. comb. with B_w

$$f(v_k) = \sum_{i=1}^{m} a_{ik} w_i$$

we associate to f a matrix $A_{B_v B_w} = (a_{ik})_{\substack{i=1,\dots,m \\ k=1,\dots,n}}$ $\Rightarrow$ $m \times n$ matrix (m rows, n columns)

with usual matrix mult.: $\underbrace{(f(v_1), \dots, f(v_n))}_{1 \times n} = \underbrace{(w_1, \dots, w_m)}_{1 \times m} \underbrace{A_{B_v, B_w}}_{m \times n}$

now:

for any vector $v = \sum_{k=1}^{n} c_k^v v_k$ we have $f(v) = \sum_{i=1}^{m} c_i^{f(v)} w_i$

$$f(v) = f\left(\sum_{k=1}^{n} c_k^v v_k\right) = \sum_{k=1}^{n} c_k^v f(v_k) = \sum_{k=1}^{n} c_k^v \sum_{i=1}^{m} a_{ik} w_i = \sum_{i=1}^{m} c_i^{f(v)} w_i$$

$$= \sum_{i=1}^{m} \underbrace{\left(\sum_{k=1}^{n} a_{ik} c_k^v\right)}_{= c_i^{f(v)}} w_i$$

$$\Rightarrow c_i^{f(v)} = \sum_{k=1}^{n} a_{ik} c_k^v$$

or $\underbrace{\begin{pmatrix} c_1^{f(v)} \\ \vdots \\ c_m^{f(v)} \end{pmatrix}}_{m\times 1} = \underbrace{A_{B_v,B_w}}_{m\times n} \cdot \underbrace{\begin{pmatrix} c_1^v \\ \vdots \\ c_n^v \end{pmatrix}}_{n\times 1}$ or $\vec{c^{f(v)}} = A_{B_v,B_w} \vec{c^v}$

addition and mult. by scalar clear

Compositions:

let $f \in \mathcal{L}(V,W)$, $g \in \mathcal{L}(W,X)$, bases B_v, B_w, B_x

$f \leftrightarrow A_{B_v,B_w} = (a_{ik})$, $g \leftrightarrow B_{B_w,B_x} = (b_{ei})$, $g\circ f \leftrightarrow C_{B_v,B_x} = (c_{ek})$

$$(g\circ f)(v_k) = g(f(v_k)) = g\left(\sum_i a_{ik} w_i\right) = \sum_i a_{ik}\, g(w_i)$$

$$= \sum_i a_{ik} \sum_e b_{ei} x_e$$

$$= \sum_e \left(\sum_i b_{ei} a_{ik}\right) x_e$$

$$= \sum_e c_{ek} x_e \qquad \Rightarrow c_{ek} = \sum_i b_{ei} a_{ik} \text{ or } C = B\cdot A$$

$\Rightarrow$ composition of lin. maps corresponds to matrix multiplication

Basis Change

vector in different bases:

Let $B_V = \{v_1, \dots, v_n\}$ and $B'_V = \{v'_1, \dots, v'_n\}$ be bases of V

we can write any $v = \sum_{i=1}^{n} c_i^{B_V} v_i = \sum_{k=1}^{n} c_k^{B'_V} v_k'$

let $v_k' = \sum_{i=1}^{n} (T_{B_V B'_V})_{ik} v_i$

$\Rightarrow v = \sum_i c_i^{B_V} v_i = \sum_k c_k^{B'_V} v_k' = \sum_k c_k^{B'_V} \sum_i (T_{B_V B'_V})_{ik} v_i$

$\Rightarrow c_i^{B_V} = \sum_k \left(T_{B_V B'_V}\right)_{ik} c_k^{B'_V}$ or $\vec{c^{B_V}} = T_{B_V B'_V} \vec{c^{B'_V}}$

We can do it the other way around, $\vec{c^{B'_V}} = T^{-1}_{B_V B'_V} \vec{c^{B_V}}$, i.e., $T_{B_V B'_V}$ is invertible.

matrix in different bases:

$f \in \mathcal{L}(V, W)$, choose bases B_V and B'_V of V and bases B_W and B'_W of W.

Let $T_{B_V B'_V}$ be matrix of basis change B_V to B'_V, $T_{B_W B'_W}$ of B_W to B'_W.

Let $A_{B_V B_W}$ be matrix of f in bases B_V, B_W, and $A_{B'_V B'_W}$ in bases B'_V, B'_W.

$f(v_k) = \sum_e (A_{B_V B_W})_{ek} w_e$

$= f\left(\sum_i \left(T^{-1}_{B_V B'_V}\right)_{ik} v_i'\right) = \sum_i (T^{-1}_{B_V B'_V})_{ik} f(v_i')$

$= \sum_i (T^{-1}_{B_V B'_V})_{ik} \sum_j \left(A_{B'_V B'_W}\right)_{ji} w_j'$

$$= \sum_{i,j} (T^{-1}_{B_v B_v'})_{ik} (A_{B_v' B_w'})_{ji} \sum_e (T_{B_w B_w'})_{ej} w_e$$

$$= \sum_e \underbrace{\sum_{i,j} (T_{B_w B_w'})_{ej} (A_{B_v' B_w'})_{ji} (T^{-1}_{B_v B_v'})_{ik}}_{= (A_{B_v B_w})_{ek}} w_e$$

$$A_{B_v B_w} = T_{B_w B_w'} A_{B_v' B_w'} T^{-1}_{B_v B_v'}$$

or $$A_{B_v' B_w'} = T^{-1}_{B_w B_w'} A_{B_v B_w} T_{B_v B_v'}$$

If $V = W$, i.e., $B_v = B_w$, $B_v' = B_w'$, we have $T_{B_v B_v'} = T_{B_w B_w'} = T$

$\Rightarrow$ $A_{B_v' B_v'} = T^{-1} A_{B_v B_v} T$, this is called conjugation by T.

This allows us to def. fct. s ϕ which are def. via A_{B_v}, but are actually indep. of basis choice, i.e., $\Phi(f) = \Phi(A^f_{B_v}) = \Phi(A^f_{B_v'})$.

Such Φ are called invariants.

two important examples for $f \in \mathcal{L}(V,V)$:

- trace: we def. $\operatorname{tr} f := \operatorname{tr} A^f_{B_v} := \sum_i (A^f_{B_v})_{ii}$

  (sum of diagonal entries for some basis B_v)

  this makes sense, since $\operatorname{tr} A^f_{B_v'} = \operatorname{tr}(T^{-1} A^f_{B_v} T) = \operatorname{tr} T T^{-1} A^f_{B_v} = \operatorname{tr} A^f_{B_v}$

(recall: $\operatorname{tr} AB = \sum_{i,j} a_{ij} b_{ji} = \sum_{i,j} b_{ji} a_{ij} = \operatorname{tr} BA$)

- determinant: we def. $\det f := \det A^f_{B_V}$ $\longrightarrow$ recall from last semester...

this makes sense since $\det A^f_{B_V'} = \det T^{-1} A^f_{B_V} T$

$$= \underbrace{\det T^{-1}}_{= \frac{1}{\det T}} \det A^f_{B_V} \det T = \det A^f_{B_V}$$

I.6 Sums and Direct Sums

<u>Def.:</u> Let $V_1, \dots, V_n$ be subsets of vector space V. Then their sum is

$$\sum_{i=1}^n V_i = V_1 + V_2 + \dots + V_n = \left\{ \sum_{i=1}^n v_i \in V, \text{ s.t. } v_i \in V_i, i = 1, \dots, n \right\}.$$

If $V_1, \dots, V_n$ are subspaces of V, then $\sum_{i=1}^n V_i = \left\{ \sum_{i=1}^n c_i v_i, \text{ s.t. } v_i \in V_i, c_i \in F \right\}$

$$= \operatorname{Span}(V_1 \cup V_2 \cup \dots \cup V_n)$$

<u>Def.:</u> Let $V = \sum_{i=1}^n V_i$. If additionally every $v \in V$ can be uniquely written as $v = v_1 + v_2 + \dots + v_n$ with $v_i \in V_i$ for all $i = 1, \dots, n$, then we call the sum a direct sum and write it as $\bigoplus_{i=1}^n V_i$.

note: for $n=2$ this is equivalent to $V_1 \cap V_2 = \{0\}$.

<u>Def.:</u> Let $V_1, \dots, V_n$ be vector spaces. Then external direct sum

$V = \bigoplus_{i=1}^n V_i$ is def. by:
- $(v_1, \dots, v_n) \in V$, $v_i \in V_i$
- $c(v_1, \dots, v_n) + c'(v_1', \dots, v_n') = (cv_1 + c'v_1', \dots, cv_n + c'v_n')$

Thm.: Let V_1, V_2 be subspaces of V, $\dim V < \infty$. Then

$$\dim(V_1 \cap V_2) + \dim(V_1 + V_2) = \dim V_1 + \dim V_2 .$$

Proof: Let $m = \dim(V_1 \cap V_2)$, choose basis $\{v_1, \dots, v_m\}$ of $V_1 \cap V_2$

$n = \dim V_1$, choose basis $\{v_1, \dots, v_m, v'_{m+1}, \dots, v'_n\}$ of V_1

$p = \dim V_2$, choose basis $\{v_1, \dots, v_m, v''_{m+1}, \dots, v''_p\}$ of V_2

claim: $\{v_1, \dots, v_m, v'_{m+1}, \dots, v'_n, v''_{m+1}, \dots, v''_p\}$ is a basis of $V_1 + V_2$.

then $\dim(V_1 + V_2) = n + p - m = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$

Assume $p > m$ (otherwise clear).

The set above is clearly generating, since any $v \in V_1 + V_2$ can be written as sum $v_1 + v_2$ with $v_1 \in V_1$, $v_2 \in V_2$, i.e.,

$$v = \underbrace{\sum_{i=1}^{m} c_i v_i + \sum_{i=m+1}^{n} d_i v_i'}_{\in V_1} + \underbrace{\sum_{i=1}^{m} e_i v_i + \sum_{i=m+1}^{p} f_i v_i''}_{\in V_2}$$

linear independence?

suppose $\underbrace{\sum_{i=1}^{m} x_i v_i}_{A} + \underbrace{\sum_{i=m+1}^{n} y_i v_i'}_{B} + \underbrace{\sum_{i=m+1}^{p} z_i v_i''}_{C} = 0$

$\Rightarrow \underbrace{A+B}_{\in V_1} = -C \Rightarrow C \in V_1 \Rightarrow C \in V_1 \cap V_2$

$\Rightarrow C = \sum_{i=1}^{m} c_i v_i$ $\square$