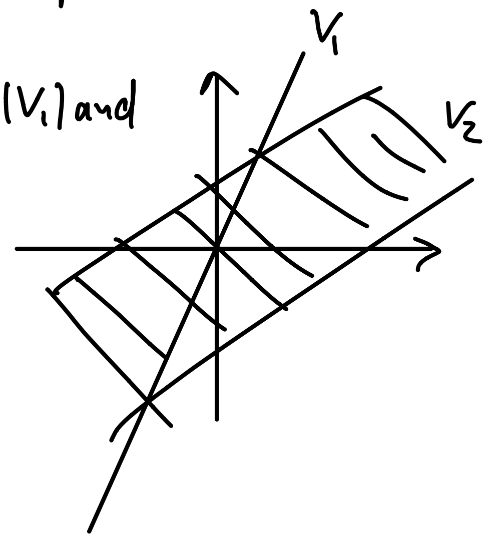
Session 8
Oct. 1, 2018

side remark: arrangements of subspaces

let $V_1, \dots, V_n \subset V$ and $V_1', \dots, V_n' \subset V$ be subspaces

we are interested in their arrangements, e.g., line (V_1) and plane (V_2) in $\mathbb{R}^3$:



$\Rightarrow$ line can be on plane ($\dim V_1 \cap V_2 = 1$) or not ($\dim V_1 \cap V_2 = 0$).

we say $V_1, \dots, V_n$ and $V_1', \dots, V_n'$ are identically arranged, if there is a linear isomorphism $f: V \to V$, s.t. $f(V_i) = V_i'$.

<u>Ex.</u>: <u>$n=1$</u>: When are V_1 and V_1' identically arranged?

need $\dim V_1 = \dim V_1'$ (since f isomorphism)

Is this enough? Yes, we can construct f by choosing basis $\{v_1, \dots, v_m\}$ of V_1 and $\{v_1', \dots, v_m'\}$ of V_1', and we extend them to bases $\{v_1, \dots, v_m, v_{m+1}, \dots, v_n\}$ of V, and $\{v_1', \dots, v_m', v_{m+1}', \dots, v_n'\}$ of V.

Then def. f by $f(v_i) = v_i'$ is an isomorphism.

$\Rightarrow V_1$ and V_1' id. arranged $\Leftrightarrow \dim V_1 = \dim V_1'$

$\underline{n=2}$: $V_1, V_2 \subset V$ and $V_1', V_2' \subset V$ subspaces

again: need $\dim V_1 = \dim V_1'$ and $\dim V_2 = \dim V_2'$

f also needs to map $V_1 \cap V_2$ into $V_1' \cap V_2'$

$\Rightarrow$ also need $\dim V_1 \cap V_2 = \dim V_1' \cap V_2'$

We say V_1 and V_2 are in general position (or intersect transversally) if $\dim V_1 \cap V_2$ is minimal (or $\dim V_1 + V_2$ maximal), given the restriction

$$\dim V_1 \cap V_2 + \dim V_1 + V_2 = \dim V_1 + \dim V_2$$

e.g., line and plane in $\mathbb{R}^3$ are in gen. pos. if they don't intersect except at 0.

↳ "most" pairs of subspaces are arranged in gen. pos.

↳ we have def. difference between zero and non-zero angle (again, for angles we need inner products)

$\underline{n=2}$: keep $\dim V_1$, $\dim V_2$, $\dim V$ fixed

- if $\dim V_1 + \dim V_2 \geq \dim V$ then V_1 and V_2 are in gen. pos. if $\dim V_1 + V_2 = \dim V$.

  $(\dim V_1 \cap V_2 \geq \dim V - \dim V_1 + V_2)$

- if $\dim V_1 + \dim V_2 < \dim V$ then V_1 and V_2 are in gen. pos. if $\dim V_1 \cap V_2 = 0$.

  $(\dim V_1 + V_2 < \dim V - \dim V_1 \cap V_2)$

Now: a few more statements about direct sums

- characterization of direct sums

sum of subspaces

<u>Thm.:</u> Let $V_1, \dots, V_n \subset V$ be subspaces with $\sum_{i=1}^n V_i = V$. Then

$$\underset{\text{direct sum}}{V = \bigoplus_{i=1}^n V_i} \iff V_{i_0} \cap \Big(\sum_{\substack{i=1 \\ i \neq i_0}}^n V_i \Big) = \{0\} \quad \forall\, 1 \leq i_0 \leq n$$

$$\iff \sum_{i=1}^n \dim V_i = \dim V$$

<u>Proof:</u> HW

- direct sums are related to projectors

<u>Def.:</u> $p \in \mathcal{L}(V)$ is a projector if $p^2 = p$ $\quad (p^2 = p \circ p)$.

Let $V = \bigoplus_{i=1}^n V_i$ be given, i.e., any v can be uniquely written as $v = \sum_{i=1}^n v_i$ with $v_i \in V_i$.

def. $p_j\big(\sum_{i=1}^n v_i\big) = v_j$, p_j is clearly linear, $p_j^2 = p_j$, $p_i p_j = 0$ for $i \neq j$,

$\sum_{i=1}^n p_i = \mathrm{id}$, and $V_i = \operatorname{im} p_i$.

This works also the other way around:

<u>Thm.:</u> Let $p_1, \dots, p_n \in \mathcal{L}(V)$ be projectors with $\sum_{i=1}^n p_i = \mathrm{id}$ and $p_i p_j = 0$ for all $i \neq j$. Then $V = \bigoplus_{i=1}^n \operatorname{im} p_i$.

<u>Proof:</u> HW

• direct sums for mappings:

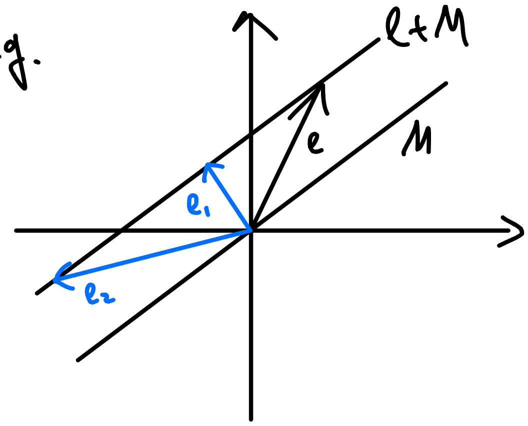
$V=\bigoplus_{i=1}^{n} V_i$, $W=\bigoplus_{i=1}^{n} W_i$, $f\in\mathcal{L}(V,W)$ s.t. $f(V_i)\subset W_i$, then we call the induced mapping $f_i: V_i \to W_i$ and we write $f=\bigoplus_{i=1}^{n} f_i$.

I.7 Quotient Spaces

let $M\subset L$ be a subspace, $\ell\in L$,

translation of M by ℓ is $\ell+M=\{\ell+m : m\in M\}$ ("linear subvariety")
"affine subset"

e.g.



note: $\ell+M$ is not a subspace unless $\ell\in M$.

Lemma: let $M_1, M_2\subset L$ be subspaces, $\ell_1,\ell_2\in L$. Then

$$\ell_1+M_1=\ell_2+M_2 \iff M_1=M_2=M \text{ and } \ell_1-\ell_2\in M$$

Proof: "$\Leftarrow$" let $m_0=\ell_1-\ell_2\in M$

$$\Rightarrow \ell_1+M=\{\ell_1+m : m\in M\},\quad \ell_2+M=\{\ell_1\underbrace{-m_0+m}_{\text{any } m'\in M} : m\in M\}$$

"$\Rightarrow$" $\ell_1+M_1=\ell_2+M_2$, $m_0=\ell_1-\ell_2$

$m_0+\underbrace{\tilde m}_{\in M_1}=0 \Rightarrow \tilde m=-m_0 \Rightarrow m_0\in M_1$

$m_0+M_1=\ell_1-\ell_2+M_1=M_2$. $0\in M_2 \Rightarrow m_0\in M_1 \Rightarrow m_0+M_1=M_1=M_2$ □