Session 9
Oct. 8, 2018

Def.: The quotient space (factor space)

L/M ("L over M", "L by M", "L mod M") is def. as

$L/M = \{ l + M : l \in L \}$ with

- addition $(l_1 + M) + (l_2 + M) = (l_1 + l_2) + M$
- scalar multiplication $c(l + M) = cl + M$

Lemma: L/M is a vector space

Remark:
- $l_1 + M = l_2 + M \iff l_1 - l_2 \in M$

  this def. an equivalence relation $\sim$ (reflexive $l \sim l$; symmetric $l_1 \sim l_2 \Rightarrow l_2 \sim l_1$; transitive $l_1 \sim l_2, l_2 \sim l_3 \Rightarrow l_1 \sim l_3$)

  $l_1 \sim l_2 \iff l_1 - l_2 \in M.$

  $\frac{L}{M} = \{$equivalence classes of $l \in L\}$

- L/M is not a subspace of L

Proof of lemma: check uniqueness of add. and sc. mult.

- If $l_1 + M = l_1' + M$ and $l_2 + M = l_2' + M$, then $l_1 + l_2 + M = l_1' + l_2' + M$,

since $l_1 + l_2 + M = l_1' + l_2' + \underbrace{l_1 - l_1'}_{\in M} + \underbrace{l_2 - l_2'}_{\in M} + M = l_1' + l_2' + M$

- similar for sc. mult
- vector space axioms checked straightforwardly

note: there is a canonical map $q: L \to L/M$, $\ell \mapsto q(\ell) = \ell + M$,

called the quotient map

↳ surjective

↳ inverse image (fiber) of $\hat{\ell} + M$ is $q^{-1}(\overbrace{\hat{\ell}+M}^{\in L/M}) = \{\ell \in L : \hat{\ell} + M = \ell + M\}$

$$= \{\ell \in L : \ell - \hat{\ell} \in M\}$$

$$= \underbrace{\hat{\ell} + M}_{\text{subset of } L}$$

↳ linear

↳ $\ker q = M$

<u>Thm.</u>: For $\dim L < \infty$, we have $\dim L/M = \dim L - \dim M$

(called the codimension of M in L).

<u>Proof</u>: $\dim L \underset{\substack{\uparrow \\ \text{rank-nullity thm. applied to } q}}{=} \dim \operatorname{im} q + \dim \ker q = \dim L/M + \dim M$. $\square$

I.8 The Fundamental Spaces of a Linear Operator

Given $f: L \to M$ (linear) we have def. $\ker f$ and $\operatorname{im} f$.

Two more interesting spaces are $\operatorname{coim} f := L/\ker f$ (coimage of f)

$\operatorname{coker} f := M/\operatorname{im} f$ (cokernel of f)

chain of mappings:

$$\begin{tikzcd} \ker f \arrow[r, "\text{canonical (just subset)}"'] & L \arrow[r, "q", "\text{canonical}"'] \arrow[rr, bend left, "f"] & \operatorname{coim} f \arrow[r, bend right, "?"'] & \operatorname{im} f \arrow[r, "\text{can.}"'] & M \arrow[r, "q"] & \operatorname{coker} f \end{tikzcd}$$

and

What about $\operatorname{coim} f \longrightarrow \operatorname{im} f$?

gen. question: given

$$\begin{tikzcd} & L \arrow[dl, "f"'] \arrow[dr, "g"] & \\ M & & N \end{tikzcd}$$

does there exist h with $hf = g$, i.e.

$$\begin{tikzcd} & L \arrow[dl, "f"'] \arrow[dr, "g"] & \\ M \arrow[rr, "h"'] & & N \end{tikzcd}$$

<u>Lemma (universal property):</u> h exists iff $\ker f \subset \ker g$. If additionally $\operatorname{im} f = M$, then h is unique.

<u>Proof:</u> "$\Rightarrow$": $g(\ell) = h(f(\ell)) = 0$ if $f(\ell) = 0$, i.e., $\ker f \subset \ker g$.

"$\Leftarrow$": construct h on $\operatorname{im} f$. Then we can extend h to all of M by choosing a basis of $\operatorname{im} f$, extend it to basis of M, set $h(e_i) = 0$ for all basis vectors e_i in the extension.

need $h(m) := g(\ell)$ if $m = f(\ell)$. Is this unique and linear?

Uniqueness: if $f(\ell_1) = m = f(\ell_2) \Rightarrow \ell_1 - \ell_2 \in \ker f \subset \ker g \Rightarrow g(\ell_1) = g(\ell_2)$.
Linearity: clear from lin. of f and g. $\square$

note:

$$\begin{tikzcd} & L \arrow[dl, "q"'] \arrow[dr, "f"] & \\ L/M \arrow[rr, "h"'] & & M \end{tikzcd}$$

h exists if $M = \ker q \subset \ker f$. If so, then it is unique because q is surjective.

complete chain from above:

$$\begin{tikzcd} \ker f \arrow[r] & L \arrow[r, "q"] \arrow[rr, bend left, "f"] & \operatorname{coim} f \arrow[r, "h"] & \operatorname{im} f \arrow[r] & M \arrow[r, "q"] & \operatorname{coker} f \end{tikzcd}$$

$$\operatorname{coim} f = L/\ker f$$

Fredholm Alternative (finite dim.):

The index of f is def. as $\operatorname{ind} f = \dim \operatorname{coker} f - \dim \ker f$

in finite dim.: $\operatorname{ind} f = \dim M/\operatorname{im} f - \dim \ker f$

$$= \dim M - \dim \operatorname{im} f - \dim \ker f$$
$$= \dim M - \dim L$$

If $\operatorname{ind} f = 0$ (e.g., $M = L$), we have the Fredholm alternative:

next time...