

Def.: The quotient space (factor space)

L/M ("Lower M ", "L by M ", "L mod M ") is def. as

$$L/M = \{ l+M : l \in L \} \text{ with}$$

- addition $(l_1+M) + (l_2+M) = (l_1+l_2)+M$
- scalar multiplication $c(l+M) = cl+M$

Lemma: L/M is a vector space

Remark: • $l_1+M = l_2+M \iff l_1-l_2 \in M$

this def. an equivalence relation \sim (reflexive $l \sim l$; symmetric $l_1 \sim l_2 \implies l_2 \sim l_1$; transitive $l_1 \sim l_2, l_2 \sim l_3 \implies l_1 \sim l_3$)

$$l_1 \sim l_2 \iff l_1 - l_2 \in M.$$

$$L/M = \{ \text{equivalence classes of } l \in L \}$$

- L/M is not a subspace of L

Proof of lemma: check uniqueness of add. and sc. mult.

• If $l_1+M = l_1'+M$ and $l_2+M = l_2'+M$, then $l_1+l_2+M = l_1'+l_2'+M$,

$$\text{since } l_1+l_2+M = l_1'+l_2' + \underbrace{l_1-l_1'}_{\in M} + \underbrace{l_2-l_2'}_{\in M} + M = l_1'+l_2'+M$$

- similar for sc. mult
- vector space axioms checked straight forwardly

note: there is a canonical map $q: L \rightarrow \frac{L}{M}$, $l \mapsto q(l) = l + M$,
called the quotient map

↳ surjective

↳ inverse image (fiber) of $\tilde{l} + M$ is $q^{-1}(\tilde{l} + M) = \{l \in L: \tilde{l} + M = l + M\}$
 $= \{l \in L: l - \tilde{l} \in M\}$
 $= \underbrace{\tilde{l} + M}_{\text{subset of } L}$

↳ linear

↳ $\ker q = M$

Thm.: For $\dim L < \infty$, we have $\dim \frac{L}{M} = \dim L - \dim M$

(called the codimension of M in L).

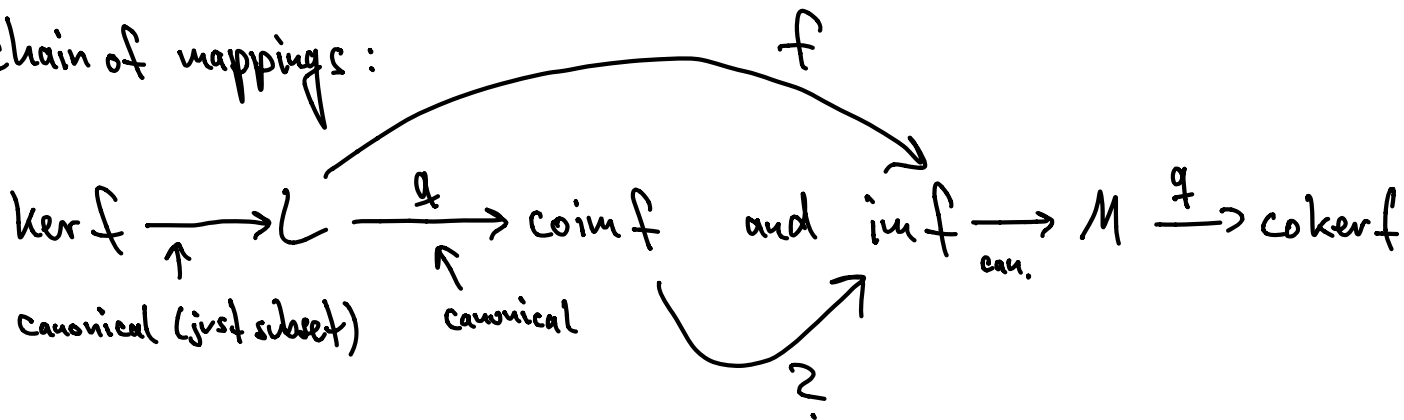
Proof: $\dim L = \dim \operatorname{im} q + \dim \ker q = \dim \frac{L}{M} + \dim M$. \square
 \uparrow
rank-nullity thm. applied to q

I.8 The Fundamental Spaces of a linear Operator

Given $f: L \rightarrow M$ (linear) we have def. $\ker f$ and $\operatorname{im} f$.

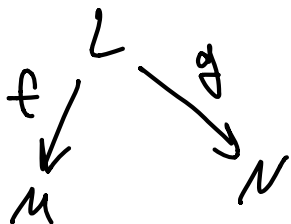
Two more interesting spaces are $\operatorname{coim} f := \frac{L}{\ker f}$ (coimage of f)
 $\operatorname{coker} f := \frac{M}{\operatorname{im} f}$ (cokernel of f)

chain of mappings:

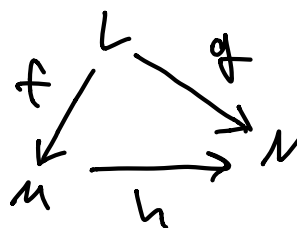


What about $\text{coim } f \rightarrow \text{im } f$?

gen. question: given



does there exist h with $h \circ f = g$, i.e.



lemma (universal property): h exists iff $\ker f \subset \ker g$. If additionally $\text{im } f = M$, then h is unique.

Proof: " \Rightarrow ": $g(l) = h(f(l)) = 0$ if $f(l) = 0$, i.e., $\ker f \subset \ker g$.

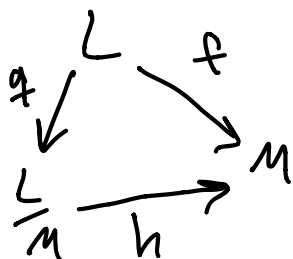
" \Leftarrow ": construct h on $\text{im } f$. Then we can extend h to all of M by choosing a basis of $\text{im } f$, extend it to basis of M , set $h(e_i) = 0$ for all basis vectors e_i in the extension.

need $h(m) := g(l)$ if $m = f(l)$. (Is this unique and linear?)

Uniqueness: if $f(l_1) = m = f(l_2) \Rightarrow l_1 - l_2 \in \ker f \subset \ker g \Rightarrow g(l_1) = g(l_2)$.

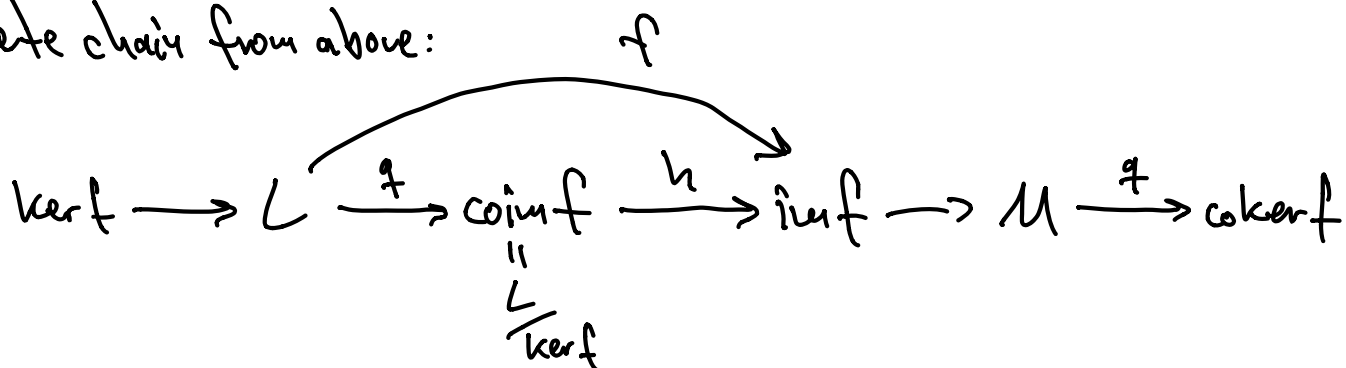
Linearity: clear from lin. of f and g . □

note:



h exists if $M = \ker g \subset \ker f$. If so, then it is unique because g is surjective.

complete chain from above:



Fredholm Alternative (finite dim.):

The index of f is def. as $\operatorname{ind} f = \dim \operatorname{coker} f - \dim \ker f$

in finite dim.: $\operatorname{ind} f = \dim \frac{M}{\operatorname{im} f} - \dim \ker f$

$$= \dim M - \dim \operatorname{im} f - \dim \ker f$$

$$= \dim M - \dim L$$

If $\operatorname{ind} f = 0$ (e.g., $M = L$), we have the Fredholm alternative:

next time...