Session 10
Oct. 10, 2018

The four fundamental spaces of a lin. map f are:

$\ker f$, $\mathrm{im} f$, $\ker f^*$, $\mathrm{im} f^*$ (f^* the dual map)

$$\ker f \subset L \xrightarrow{f} M \supset \mathrm{im} f$$

$$\mathrm{im} f^* \subset L^* \xleftarrow{f^*} M^* \supset \ker f^*$$

<u>Def.:</u> Let $M \subset L$ be a subspace. Then $M^\perp \subset L^*$, the orthogonal complement of M (Axler: "annihilator of M"), is def. as $\{m^* \in L^* : m^*(m) = 0 \ \forall m \in M\}$.

note: $M^\perp$ is a subspace

big diagram relating all involved spaces ($\dim L < \infty$, $\dim M < \infty$)
($\cong$ means canonically isomorphic)

$$\begin{tikzcd}
(\mathrm{im} f^*)^* \cong \mathrm{coim} f =: L/\ker f & M/\mathrm{im} f := \mathrm{coker} f \cong (\ker f^*)^* \\
(\mathrm{coker} f^*)^* \cong (\mathrm{im} f^*)^\perp \cong \ker f \subset L \arrow[u, "q"'] \arrow[r, "f"] & M \supset \mathrm{im} f \cong (\ker f^*)^\perp \cong (\mathrm{coim} f^*)^* \arrow[u, "q"] \\
(\mathrm{coim} f)^* \cong (\ker f)^\perp = \mathrm{im} f^* \subset L^* \arrow[d, "q"'] & M^* \supset \ker f^* = (\mathrm{im} f)^\perp \cong (\mathrm{coker} f)^* \arrow[l, "f^*"'] \arrow[d, "q"] \\
(\ker f)^* \cong \mathrm{coker} f^* =: L^*/\mathrm{im} f^* & M^*/\ker f^* := \mathrm{coim} f^* \cong (\mathrm{im} f)^*
\end{tikzcd}$$

for the proof we need two lemmas:

<u>Lemma 1</u>: For M a subspace of L, $\frac{L^*}{M^\perp} \cong M^*$ (can. isom.)

<u>Lemma 2</u>: For M a subspace of L, $\left(\frac{L}{M}\right)^* \cong M^\perp$ (can. isom.)

<u>Proof of lemma 1:</u>

def. $i: \frac{L^*}{M^\perp} \longrightarrow M^*$, $\underset{\in L^*}{f} + M^\perp \longmapsto f|_M$ (restriction of f to M)

↳ i is linear

↳ i is surjective, clear

↳ i is injective, since $\ker i = \{0\}$ ($f|_M = 0 \Rightarrow f \in M^\perp$ and $f + M^\perp = M^\perp$ is the zero element of $\frac{L^*}{M^\perp}$) $\square$

<u>Proof of lemma 2:</u>

first, note that $\dim\left(\frac{L}{M}\right)^* = \dim \frac{L}{M} = \dim L - \dim M \overset{\text{see HW}}{=} \dim M^\perp$

construct canonical isomorphism

quotient map $q: L \longrightarrow L/M$, dual map $q^*: \left(\frac{L}{M}\right)^* \to L^*$

we show $\ker q^* = \{0\}$ and $\operatorname{im} q^* = M^\perp$

- $q^*: \left(\frac{L}{M}\right)^* \to L^*$, $g \longmapsto q^*(g)$, i.e., $q^*(g)(\ell) = g(q(\ell)) = g(\ell + M)$

so $q^*(g) = 0$ means $q^*(g)(\ell) = 0 \ \forall \ell \in L$, i.e. $g(\ell + M) = 0 \ \forall \ell \in L$

so $g = 0$

$\Rightarrow \ker q^* = \{0\}$

- let $\ell' \in \operatorname{im} q^*$, then for $m \in M$ then $\ell'(m) = q^*(g)(m)$ for some $g \in \left(\frac{L}{M}\right)^*$

$$= g(m+M)$$
$$= g(M) = 0$$

so $\operatorname{im} q^* \subset M^\perp$

let $m' \in M^\perp$, just def. $\tilde{m} \in \left(\frac{L}{M}\right)^*$ by

$\tilde{m} : \frac{L}{M} \to \mathbb{C}, \ \ell + M \mapsto \tilde{m}(\ell+M) := m'(\ell+M) = m'(\ell)$

$\Rightarrow q^*(\tilde{m}) = m'$, so $M^\perp \subset \operatorname{im} q^* \Rightarrow \operatorname{im} q^* = M^\perp$

$\Rightarrow q^* : \left(\frac{L}{M}\right)^* \longrightarrow M^\perp$ is a (canonical) isomorphism $\square$