I.9 The Structure of Linear Operators in Finite Dimension

For this chapter: L, M are finite dim. vector spaces

Goal: understand structure of lin. maps $L \to M$, e.g., $L_0 \oplus L_1 \xrightarrow{f} M_1 \oplus M_2$

or: Is there a basis s.t. matrix of lin. map has a particularly simple form?

↳ when L, M are unrelated, freedom in basis choice makes question easy (next thm.)
↳ more interesting: $f: L \to L$ and consider only one basis in L

Easy part:

<u>Thm.</u>: Let $f \in \mathcal{L}(L,M)$. Then there are $\tilde{L} \subset L$ and $M \subset M'$ such that

$L = \ker f \oplus \tilde{L}$, $M = \operatorname{im} f \oplus M'$ and $\tilde{f}: \tilde{L} \to \operatorname{im} f$, $\tilde{f} = f|_{\tilde{L}}$ is an isomorphism.
Also, there are bases of L and M such that the matrix A_f of f is

$$A_f = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}, \quad E_r = \mathrm{id}_r = r \times r \text{ identity matrix, with } r = \dim \operatorname{im} f =: \operatorname{rank} f$$

$$\left((A_f)_{ij} = \begin{cases} \delta_{ij} & \text{for } 1 \leq i,j \leq r \\ 0 & \text{else} \end{cases} \right)$$

<u>Proof</u>: $\tilde{L}$ such that $L = \ker f \oplus \tilde{L}$ and $\tilde{M}$ s.t. $M = \underbrace{\operatorname{im} f}_{=: \tilde{M}} \oplus M'$ exist due to projector thm. (see I.6), or by choosing and extending basis of $\operatorname{im} f$.

$\tilde{f}: \tilde{L} \to \tilde{M}$, $\tilde{f} = f|_{\tilde{L}}$ is clearly injective ($\ker f \cap \tilde{L} = \{0\}$, so $\ker \tilde{f} = \{0\}$).

$\tilde{f}$ is also clearly surjective

$$\left(\tilde{m} \in \operatorname{im} f \Rightarrow f(\ell) = \tilde{m},\ \ell = \underset{\ker f}{\ell_0} + \underset{\tilde{L}}{\hat{\ell}} \Rightarrow \tilde{m} = f(\ell_0 + \hat{\ell}) = f(\hat{\ell}) = \tilde{f}(\hat{\ell})\right)$$

$\Rightarrow \tilde{f}$ is an isomorphism

let $r = \dim \tilde{L} = \dim \tilde{M}$, choose basis $\{\underbrace{e_1, \dots, e_r}_{\text{basis of } \tilde{L}}, \underbrace{e_{r+1}, \dots, e_n}_{\text{basis of } \ker f}\}$ of L

$\Rightarrow \{f(e_1), \dots, f(e_r)\}$ basis of $\tilde{M}$, extend it to basis of M. $\square$

note: this means any $m \times n$ matrix can be brought into form $\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}$ by basis change, i.e., $\exists$ invertible (non-singular) matrices B, C s.t. $\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} = BAC$

(see I.5)

now: $f \in \mathcal{L}(L) = \mathcal{L}(L, L)$

want: $L = \underset{\circlearrowleft f}{L_1} \oplus \underset{\circlearrowleft f}{L_2} \oplus \dots \oplus \underset{\circlearrowleft f}{L_n}$, then each $f|_{L_i}$ is easier to handle

most simple and also typical case: $\dim L_i = 1$, then f is called diagonalizable

Terminology:

<u>Def.:</u> let $\tilde{L} \subset L$ be a subspace, $f \in \mathcal{L}(L)$

- $\tilde{L}$ is called invariant if $f(\tilde{L}) \subset \tilde{L}$ $(\tilde{\ell} \in \tilde{L} \Rightarrow f(\tilde{\ell}) \in \tilde{L})$.
- If $\tilde{L}$ is invariant and $\dim \tilde{L} = 1$, then $\tilde{L}$ is called a proper subspace (for f).

Then $f|_{\tilde{L}}(\ell) = \lambda \ell$ for some $\lambda \in F$, or $f|_{\tilde{L}} = \lambda\, \underset{\text{identity on } \tilde{L}}{\mathrm{id}_{\tilde{L}}}$, i.e. $f|_{\tilde{L}}$ is multiplication by a constant. λ is called eigenvalue of f and any $\tilde{L} \ni \tilde{e} \neq 0$ is called eigenvector.

- If $\tilde{L}$ is invariant and $f|_{\tilde{L}}$ is multiplication by a constant, then $\tilde{L}$ is called eigenspace.

<u>Def.:</u> $f \in \mathcal{L}(L)$ is called diagonalizable if $L = \bigoplus_{i=1}^{n} L_i$ with proper subspaces L_i of f (i.e., $\dim L_i = 1 \; \forall i$)

note: this is equivalent to existence of basis such that matrix of f is diagonalizable:

"$\Leftarrow$" A_f diag. in basis $\{e_1, \dots, e_n\} \Rightarrow f(e_i) = \lambda_i e_i \Rightarrow \operatorname{span}\{e_i\}$ proper subspaces.

"$\Rightarrow$" choose any $e_i \in L_i \Rightarrow$ basis for A_f.

<u>Def.:</u> For $f \in \mathcal{L}(L)$, we call $P_f(t) = \det(t \cdot id - f)$ the characteristic polynomial of f.

note: • $\det(t \cdot id - B^{-1} A_f B) = \det(B^{-1}(t \cdot id - A_f) B)$

$$= (\det B)^{-1} \det(t \cdot id - A_f) \det B$$

so char. pol. can be computed with matrix of f, and it is indep. of basis choice

- $P_f(t) = t^n - \operatorname{tr} f \, t^{n-1} + \dots + (-1)^n \det f$

- Don't be afraid of determinants (like Axler).

Remember: det ~ oriented volume (then formal def. makes sense)

Thm.: λ eigenvalue of $f \iff \lambda$ root of $P_f(t)$ in $\mathbb{F}$.

Proof:

"$\Leftarrow$" let $0 = P_f(\lambda) = \det(\lambda \mathrm{id} - f)$

Then $\dim \operatorname{im}(\lambda \mathrm{id} - f) \neq \dim L$ ($f \in \mathcal{L}(L)$), so $\dim \ker(\lambda \mathrm{id} - f) \neq 0$

let $0 \neq e \in \ker(\lambda \mathrm{id} - f)$, then $(\lambda \mathrm{id} - f)(e) = 0$ or

$f(e) = \lambda e \Rightarrow \lambda$ eigenvalue to eigenvector e.

"$\Rightarrow$" If $f(e) = \lambda e$ $(e \neq 0) \Rightarrow e \in \ker(\lambda \mathrm{id} - f) \Rightarrow \det(\lambda \mathrm{id} - f) = 0$

note: • not all f are diagonalizable

• if $\mathbb{F} = \mathbb{R}$, $P_f(t)$ might not have roots in $\mathbb{R}$ at all ($\mathbb{R}$ not algebraically closed)

• if $\mathbb{F} = \mathbb{C}$, $P_f(t) = \prod_{i=1}^{n} (t - \lambda_i)^{r_i}$ has at least one root ($\mathbb{C}$ algebraically closed), i.e., $\exists$ at least one proper subspace

r_i = multiplicity of the root

$\sigma(f) = \{\lambda \in \mathbb{C} : P_f(\lambda) = 0\}$ = spectrum of f

If all $r_i = 1$, then spectrum is called simple