Session 12
Oct. 17, 2018

Ex.: $\dim L = 2$, consider matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $f \in \mathcal{L}(L)$

$$\det(t\,\mathrm{id} - A) = t^2 - t\cdot \mathrm{tr}A + \det A$$
$$= t^2 - (a+d)t + (ad - bc)$$

$$\Rightarrow \text{roots } \lambda_{\pm} = \frac{a+d}{2} \pm \sqrt{\left(\frac{a+d}{2}\right)^2 - ad + bc}$$
$$= \frac{a+d}{2} \pm \sqrt{\left(\frac{a-d}{2}\right)^2 + bc}$$

- $\lambda_+ \neq \lambda_-$: let e_+, e_- be such that $f(e_\pm) = \lambda_\pm e_\pm$, i.e., $\mathrm{span}\{e_+\}$, $\mathrm{span}\{e_-\}$ are proper subspaces of f. Is $L = \mathrm{span}\{e_+\} \oplus \mathrm{span}\{e_-\}$?

  Let $c_1 e_+ + c_2 e_- = 0$. Then
  $$0 = \lambda_+(c_1 e_+ + c_2 e_-) - f(c_1 e_+ + c_2 e_-)$$
  $$= \lambda_+(c_1 e_+ + c_2 e_-) - (c_1 \lambda_+ e_+ + c_2 \lambda_- e_-)$$
  $$= \underbrace{(\lambda_+ - \lambda_-)}_{\neq 0} c_2 e_-, \text{ so } c_2 = 0. \text{ Similar for } c_1.$$

  $\Rightarrow$ e_+ and e_- linearly independent, so f diagonalizable (A_f diagonal in basis $\{e_+, e_-\}$)

- $\lambda_+ = \lambda_- = \lambda$, i.e., $(a-d)^2 + 4bc = 0$

  f diagonalizable only if $f(e) = \lambda e$ for all $e \in L$, i.e., $a = d = \lambda$, $b = c = 0$,

  or $A = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in any basis ($L = \mathrm{span}\{e_1\} \oplus \mathrm{span}\{e_2\}$ for any two lin. indep. e_1, e_2)

  otherwise f not diagonalizable ($L_\lambda = \ker(f - \lambda\,\mathrm{id})$ has $\dim L_\lambda = 1 \neq \dim L$)

Ex.: $\begin{pmatrix}\lambda & b\\ 0 & \lambda\end{pmatrix}$, $\lambda \neq 0$, $b \neq 0$ $\Rightarrow P(t) = (t-\lambda)^2$, $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \lambda\begin{pmatrix}x\\ y\end{pmatrix}$

$\Rightarrow \lambda x + y = \lambda x$, $\lambda y = \lambda y \Rightarrow y = 0$

$\Rightarrow \lambda$ is eigenvalue on proper subspace span$\left\{\begin{pmatrix}1\\ 0\end{pmatrix}\right\}$

Remarks: • f is typically diagonalizable (on $\mathbb{C}$);
if not "small change" (say, of matrix elements) makes it diagonalizable
• it is generally true that when all eigenvalues are pairwise different, then f is diagonalizable

now: taking fct.s of f (here: polynomials only)

If $Q(t) = \sum_{i=0}^{n} c_i t^i$, then $Q(f) = \sum_{i=0}^{n} c_i f^i$, $f^i = \underbrace{f \circ f \circ \ldots \circ f}_{i \text{ times}}$

Def.: • A polynomial Q annihilates f if $Q(f) = 0$.
• $Q(t) = \sum_{i=0}^{n} c_i t^i$ with $c_n = 1$ and minimal n s.t. still $Q(f) = 0$ is called minimal polynomial.

note: • $\dim \mathcal{L}(L) = (\dim L)^2$, so $id, f, f^2, \ldots, f^{n^2}$ linearly dependent, so there is a polynomial with degree $\leq n^2$ which annihilates f
• minimal polynomial is unique: Q_1, Q_2 minimal $\Rightarrow Q_1 - Q_2$ annihilates f but has lower degree

Thm. (Cayley-Hamilton): $P_f(f) = 0$

Proof (for $F = \mathbb{C}$ only, also true if F not alg. closed):

For $\dim L = 1$, $f(l) = \lambda l$ $\forall l \in L$, so $P_f(t) = (t - \lambda)$ and $P_f(f) = 0$.
Let $\dim L = n \geq 2$.

Take eigenvector e_1 to eigenvalue λ, $L = \mathrm{span}\{e_1\}$ eigenspace.

Let $\{e_1, e_2, \dots, e_n\}$ be basis of L.

Consider $\bar{f}: \frac{L}{L_1} \to \frac{L}{L_1}$, $\bar{f}(e + L_1) = f(e) + L_1$, then $\bar{e}_i = e_i + L_1$ $(i \geq 2)$ basis of L/L_1.

$$\text{In these bases } A_f = \begin{pmatrix} \lambda & \overbrace{\cdots\cdots}^{\text{some values}} \\ 0 & \\ \vdots & A_{\bar{f}} \\ 0 & \end{pmatrix}$$

$$\Rightarrow P_f(t) = \det(t\,\mathrm{id} - A_f) \underset{\substack{\uparrow \\ \text{recall computation rules} \\ \text{for determinant}}}{=} (t-\lambda)\det(t\,\mathrm{id} - A_{\bar{f}}) = (t-\lambda) P_{\bar{f}}(t)$$

Now: induction in n. $n=1$ shown above.

induction step: Assume $P_{\bar{f}}(\bar{f}) = 0$.

Then $P_{\bar{f}}(f)e \in L_1$ $\forall e \in L$, so $P_f(f)e = \underbrace{(f-\lambda)}_{=0 \text{ on } L_1} \underbrace{P_{\bar{f}}(f)e}_{\in L_1} = 0$. $\square$

from now on: $f - \lambda\,\mathrm{id} = f - \lambda$

Now: get back to example $A_f = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, eigenvalue λ, proper subspace $L_\lambda = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$

we have $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, so $(f-\lambda)(e) \in L_\lambda$ $\forall e \in L$

then $(f-\lambda)\underbrace{(f-\lambda)(e)}_{\in L_\lambda} = 0$ $\forall e \in L$, so L is a "generalized eigenspace"

<u>Def.:</u>
- $e \in L$ is called a root vector of f if $(f-\lambda)^r(e) = 0$ for some $r > 0$
- If λ is additionally an eigenvalue, then such e which are non-zero

are called generalized eigenvectors

- Generalized eigenspace $G(\lambda) = \{\text{generalized eigenvectors to } \lambda\} \cup \{0\}$