Session 13
Oct. 29, 2018

note: <u>Def.:</u> $f$ is called nilpotent if $f^j = 0$ for some $j$

we will prove:

<u>Thm. (abstract Jordan decomposition):</u>
let $f \in \mathcal{L}(L)$ with $\dim L = n < \infty$, and $L$ a vector space over the algebraically closed field $\mathbb{F}$. Then $L = \bigoplus_i G(\lambda_i)$ and $f = \bigoplus_i f_i$ with $f_i : G(\lambda_i) \to G(\lambda_i)$, where the $\lambda_i$ are the distinct eigenvalues.

we prove this via a series of lemmas. Always assumed: $f \in \mathcal{L}(L)$ and $\dim L = n$.

<u>Lemma A:</u> $L = \ker f^n \oplus \operatorname{im} f^n$

<u>Proof:</u> First show that $\ker f^n \cap \operatorname{im} f^n = \{0\}$, i.e., we have a direct sum.
let $\ell \in \ker f^n \cap \operatorname{im} f^n \Rightarrow f^n(\ell) = 0$ and $\exists\, m \in L$ s.t. $\ell = f^n(m)$
$\Rightarrow f^{2n}(m) = 0$
now: $\ker f^{2n} = \ker f^{2n-1} = \dots = \ker f^n \supset \ker f^{n-1} \supset \dots \supset \ker f \supset \{0\}$
(if $\subset$ but $\neq$ then dim increases by at least 1; but once $\ker f^m = \ker f^{m+1}$, then if $v \in \ker f^{m+2}$, $f^{m+1}(f(v)) = 0$, i.e., $f(v) \in \ker f^{m+1} = \ker f^m$, so $f^{m+1}(v) = f^m(f(v)) = 0 \Rightarrow v \in \ker f^{m+1}$)
so also $f^n(m) = 0 \Rightarrow \ell = 0$
then by Fundamental Thm. of Linear Maps:

$$\dim(\ker f^n \oplus \operatorname{im} f^n) = \dim \ker f^n \oplus \dim \operatorname{im} f^n = \dim L \qquad \square$$

<u>Lemma B:</u> $G(\lambda) = \ker(f-\lambda)^n$

<u>Proof:</u> "$\supset$": let $\ell \in \ker(f-\lambda)^n \Rightarrow (f-\lambda)^n(\ell) = 0$

"$\subset$": let $\ell \in G(\lambda) \Rightarrow \ell \in \ker(f-\lambda)^j$ for some $j$, choose $j = n$ $\square$

**Lemma C:** Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues with corresponding generalized eigenvectors $\ell_1, \dots, \ell_m$. Then $\ell_1, \dots, \ell_m$ are linearly independent.

**Proof:** Let $\sum_{i=1}^{m} c_i \ell_i = 0$.

Let $(f-\lambda_1)^k \ell_1 \neq 0$ for largest possible $k$, $w := (f-\lambda_1)^k \ell_1$

so that $(f-\lambda_1) w = 0$. Then $(f-\lambda)w = (\lambda_1 - \lambda) w \quad \forall \lambda \in F$

$\Rightarrow (f-\lambda)^n w = (\lambda_1 - \lambda)^n w \quad (*)$

$$\Rightarrow 0 = (f-\lambda_1)^k (f-\lambda_2)^n \dots (f-\lambda_m)^n \left( \sum_{i=1}^{m} c_i \ell_i \right)$$

$$= c_1 (f-\lambda_2)^n \dots (f-\lambda_m)^n w \qquad \left( G(\lambda) = \ker (f-\lambda)^n \right)$$

$$\overset{\text{by } (*)}{=} c_1 (\lambda_1 - \lambda_2)^n \dots (\lambda_1 - \lambda_m)^n w$$

$\Rightarrow c_1 = 0$. Same way: $c_i = 0 \ \forall i = 1, \dots, m$ $\square$

**Lemma D:** $\ker p(f)$ and $\operatorname{im} p(f)$ are invariant for any polynomial $p$.

**Proof:**
- $\ell \in \ker p(f) \Rightarrow p(f)(\ell) = 0$
  $\Rightarrow p(f)(f(\ell)) = f(p(f)(\ell)) = f(0) = 0$, so also $f(\ell) \in \ker p(f)$
- $\ell \in \operatorname{im} p(f) \Rightarrow \exists\, m \in L$ s.t. $p(f)(m) = \ell$
  $\Rightarrow f(\ell) = f(p(f)(m)) = p(f)(f(m))$, so also $f(\ell) \in \operatorname{im} p(f)$

**Proof of abstract Jordan decomposition:**

First, note that $G(\lambda_i) = \ker(f-\lambda_i)^n \ \forall i = 1, \dots, n$ (Lemma B), so with Lemma D $G(\lambda_i)$ is invariant under $f$.

Now: prove $L = \bigoplus_i G(\lambda_i)$ by induction. $n=1$ clear, assume $n>1$ and induction hypothesis.

Field $\mathbb{F}$ algebraically closed $\Rightarrow$ there is at least one eigenvalue $\lambda_1$

Lemma A for $f-\lambda_1$ gives $L = \underbrace{\ker(f-\lambda_1)^n}_{\substack{=G(\lambda_1)\neq\{0\} \\ \text{(Lemma B)}}} \oplus \underbrace{\operatorname{im}(f-\lambda_1)^n}_{=:U}$ with $U$ invariant (Lemma D).

$\Rightarrow \dim U < n$; also $f|_U$ has eigenvalues $\lambda_2, \dots, \lambda_m$ ($m = \#$ of distinct eigenvalues)

$\Rightarrow$ by induction hypothesis $U = \bigoplus_{i=2}^{m} G(\lambda_i, f|_U)$

Finally: Is $G(\lambda_i, f|_U) = G(\lambda_i, f)$ for $i = 2, \dots, m$?

- $G(\lambda_i, f|_U) \subset G(\lambda_i, f)$ clear
- Let $\ell \in G(\lambda_i, f)$, then $\ell = \underbrace{\ell_1}_{\in G(\lambda_1, f)} + u^{\in U}$ and $u = \sum_{j=2}^{m} \ell_j$, $\ell_j \in G(\lambda_j, f|_U) \subset G(\lambda_j, f)$

$\Rightarrow \ell = \sum_{j=1}^{m} \ell_j \Rightarrow$ by Lemma C all $\ell_j = 0$ except $\ell_i$.

since $\ell_1 = 0 \Rightarrow \ell = u \in U \Rightarrow \ell \in G(\lambda_i, f|_U)$. $\square$

<u>Corollary:</u> $L$ has a basis of generalized eigenvectors.

<u>Corollary:</u> $f$ has simple spectrum $\Rightarrow f$ diagonalizable

<u>note:</u> multiplicity of eigenvalue $\lambda = \dim G(\lambda)$, so $\dim L =$ sum of multiplicities of all eigenvalues

Next: • an $r \times r$ matrix of the form $J_r(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$ is called Jordan block.

• $J = \begin{pmatrix} J_{r_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{r_m}(\lambda_m) \end{pmatrix}$ is called Jordan matrix

• Jordan basis of $f$ = basis s.t. $f$ is represented by Jordan matrix, i.e., has Jordan normal form: there is non-singular $X$ s.t. $X^{-1} A_f X = J$