Session 14
Nov. 5, 2018

We want to prove:

Thm. (Jordan normal form): Every $f \in \mathcal{L}(L)$ has a Jordan basis.

Proof: consider simple case first:

Let $f \in \mathcal{L}(L)$ have one eigenvalue $\lambda \Rightarrow (f-\lambda)^n = 0$, i.e., $g = f - \lambda$ is nilpotent and has eigenvalue 0.

We prove below that such f have a Jordan basis with zeros on diagonal.
Then the general case follows from $L = \bigoplus_i G(\lambda_i)$ with corresponding

$f_i : G(\lambda_i) \to G(\lambda_i)$, since each $(f-\lambda_i)|_{G(\lambda_i)}$ is nilpotent

$\Rightarrow f_i = f|_{G(\lambda_i)}$ has a Jordan basis (with λ_i on diagonal) $\square$

It remains to prove:

Lemma: Let $g \in \mathcal{L}(L)$ be nilpotent. Then there are $e_1, \dots, e_n \in L$ and integers $m_1, \dots, m_n > 0$ such that

$g^{m_1}(e_1), g^{m_1 - 1}(e_1), \dots, g(e_1), e_1, \dots, g^{m_n}(e_n), \dots, g(e_n), e_n$ is a basis of L

and $g^{m_i+1}(e_i) = 0 \;\; \forall i = 1, \dots, n$.

note: • Nonzero vectors $e, g(e), g^2(e), \dots, g^k(e)$, s.t. $g^{k+1}(e) = 0$ are called a string of g.

• Consider a single string basis, then write in components:

$g^k(e) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, g^{k-1}(e) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, g(e) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix}, e = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$

so matrix of g is $\begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$.

- Thus the lemma proves that every nilpotent g has a Jordan basis, possibly with several Jordan blocks: $\begin{pmatrix} J_{r_1}(0) & & & 0 \\ & J_{r_2}(0) & & \\ & & \ddots & \\ 0 & & & J_{r_n}(0) \end{pmatrix}$, e.g., $\begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} \end{pmatrix}$

Proof of lemma:

We use induction in $\dim L$.

If $\dim L = 1$ only $g = 0$ is nilpotent, lemma holds for any $0 \neq e_1 \in L$ (with $m_1 = 0$).

Let $\dim L > 1$ and assume induction hypothesis.

g nilpotent $\Rightarrow$ if λ eigenvalue then $g(v) = \lambda v \Rightarrow 0 = g^n(v) = \lambda^n v \Rightarrow \lambda = 0$, i.e., g only has one eigenvalue 0

$\Rightarrow \det g = 0 \Rightarrow g$ not injective and g not surjective

$\Rightarrow \dim \operatorname{im} g < \dim L$

Let induction hypothesis hold for $g|_{\operatorname{im}(g)}$ (note: assume $\operatorname{im}(g) \neq \{0\}$, otherwise clear), i.e.,

$g^{m_1}(e_1), \dots, g(e_1), e_1, \dots, g^{m_n}(e_n), \dots, g(e_n), e_n$ basis of L (*) and

$g^{m_1+1}(e_1) = \dots = g^{m_n+1}(e_n) = 0$ for $e_1, \dots, e_n \in \operatorname{im}(g)$

$\Rightarrow \exists v_1, \dots, v_n \in L$, s.t. $g(v_j) = e_j \ \forall j = 1, \dots, n \Rightarrow g^{k+1}(v_j) = g^k(e_j)$

Claim: $g^{m_1+1}(v_1), \dots, g(v_1), v_1, \dots, g^{m_n+1}(v_n), \dots, g(v_n), v_n$ lin. indep. (**)

- let lin.comb. of these vectors $= 0$
- then apply g => all coefficient $= 0$ except possibly those for $g^{m_j+1}(v_j) = g^{m_j}(e_j)$ due to (*)
- but these are also 0 due to (*)

Now: extend (**) to basis of L by adding vectors $w_1, \dots, w_p$

=> each $g(w_j)$ is in span of (*) and g(**) gives (*), so there is x_j in span of (**) such that $g(w_j) = g(x_j)$

=> def. $v_{n+j} = w_j - x_j$ => $g(v_{n+j}) = 0$

=> (**) with $v_{n+1}, \dots, v_{n+p}$ spans L (span contains each x_j and each v_{n+j}, i.e., w_j)

=> also a basis, since # of lin. indep. vectors = # of basis vectors

=> we have found a basis as claimed $\square$