Session 15
Nov. 7, 2018

Uses of Jordan normal form:

- solve $y'(t) = A y(t)$, $y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$, $A = n \times n$ matrix

  ↳ let $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \Rightarrow y(t) = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$

  ↳ let $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \Rightarrow \begin{array}{l} y_1'(t) = \lambda y_1(t) + y_2(t) \\ y_2'(t) = \lambda y_2(t) \end{array}$

  $\Rightarrow y_2(t) = c e^{\lambda t}$, $y_1(t) = c t e^{\lambda t}$

  HW: formula for any Jordan block

  HW: solve $y'(t) = A y(t)$ in general by changing A to Jordan normal form

- taking powers of matrix: $A = X J X^{-1}$, J = Jordan normal form

  $\Rightarrow A^k = X J X^{-1} X J X^{-1} \dots X J X^{-1} = X J^k X^{-1}$

I.10 (De)complexification

Decomplexification

L, M vector spaces over $\mathbb{C}$, $\dim L < \infty$, $\dim M < \infty$

multiplication only over $\mathbb{R}$ $\Rightarrow$ real vector space $L_{\mathbb{R}}$ (decomplexification of L)

$f: L \to M$ linear $\Rightarrow f_{\mathbb{R}}: L_{\mathbb{R}} \to M_{\mathbb{R}}$ (decomplexification of f)

L: over $\mathbb{C}$, M: over $\mathbb{C}$

Thm.: a) If $e_1,\dots,e_n$ basis of L over $\mathbb{C}$, then $e_1,\dots,e_n, ie_1,\dots,ie_n$ basis of $L_{\mathbb{R}}$ over $\mathbb{R}$ (in particular: $\dim_{\mathbb{R}} L_{\mathbb{R}} = 2\dim_{\mathbb{C}} L$).

b) $f: L \to M$ linear with matrix $A = B + iC$, B, C real matrices, in bases $e_1,\dots,e_m$ and $e_1',\dots,e_m'$, then matrix of $f_{\mathbb{R}}: L_{\mathbb{R}} \to M_{\mathbb{R}}$ is

$\begin{pmatrix} B & -C \\ C & B \end{pmatrix}$ in bases $e_1,\dots,e_m, ie_1,\dots,ie_m$ and $e_1',\dots,e_n', ie_1',\dots,ie_n'$.

c) $f: L \to L$ linear, then $\det f_{\mathbb{R}} = |\det f|^2$

Proof:

a) any $e \in L$: $e = \sum_{k=1}^m a_k e_k = \sum_{k=1}^m (b_k + ic_k) e_k = \sum_{k=1}^m b_k e_k + \sum_{k=1}^m c_k (ie_k)$

$\Rightarrow$ generating

also, if lin. comb. $= 0 \Rightarrow b_k + ic_k = 0$ ($e_1,\dots,e_m$ lin. indep.)

$\Rightarrow b_k = 0 = c_k$

b) recall def. of matrix elements:

$$(f(e_1),\dots,f(e_m)) = (e_1',\dots,e_n')(B+iC)$$

$\mathbb{C}$-linearity: $(f(ie_1),\dots,f(ie_m)) = (e_1',\dots,e_n')(-C+iB)$

$$\Rightarrow (f(e_1),\dots,f(e_m), f(ie_1),\dots,f(ie_m)) = (e_1',\dots,e_n', ie_1',\dots,ie_n')\begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

c) Let matrix of f be $B+iC$

$$\Rightarrow \det f_{\mathbb{R}} = \det\begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

$$\overset{(+i)\text{ bottom row}}{=} \det\begin{pmatrix} B+iC & -C+iB \\ C & B \end{pmatrix}$$

$$\overset{(-i)\text{ first column}}{=} \det\begin{pmatrix} B+iC & 0 \\ C & B-iC \end{pmatrix}$$

$$= \det(B+iC)\det(B-iC)$$

$$= \det f \cdot \overline{\det f}$$

$$= |\det f|^2 \qquad \square$$

Complex Structure

back from $L_{\mathbb{R}}$ to L? Need to know $J: L_{\mathbb{R}} \to L_{\mathbb{R}}$, $e \mapsto J(e) = ie$, $J^2 = -\mathrm{id}$

Def.: Let V be vector space over $\mathbb{R}$. Then lin. operator $J: V \to V$ with $J^2 = -\mathrm{id}$ is called complex structure on V.

Thm.: Let V be a real vector space with complex structure J and let $(a+ib)e = ae + bJ(e)$ be def. as multiplication with complex numbers to yield a space $\tilde{V}$. Then $\tilde{V}$ is a complex vector space, and $\tilde{V}_{\mathbb{R}} = V$.

Proof: clear, associativity in $\mathbb{C}$ follows from $J^2 = -\mathrm{id}$. $\square$

note: • $\dim_{\mathbb{R}} V = 2\dim_{\mathbb{C}} \tilde{V}$ is even

• matrix of J can be chosen (in some basis) as $\begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$, $E_n = n\times n$ unit matrix

Complexification:

Fix real vector space V. On $V \oplus V$, take $J(e_1, e_2) = (-e_2, e_1)$

$\Rightarrow J^2(e_1, e_2) = J(-e_2, e_1) = (-e_1, -e_2)$, so $J^2 = -\mathrm{id}$ is a complex structure

Complexification of V is $\widetilde{V \oplus V} =: V^{\mathbb{C}}$ with this J.

Note: • $v \in V$ identified with $(v, 0) \in V \oplus V$

$i(v, 0) = J(v, 0) = (0, v)$

$\Rightarrow V^{\mathbb{C}} \ni (e_1, e_2) = (e_1, 0) + (0, e_2) = (e_1, 0) + i(e_2, 0) = e_1 + ie_2$

$\Rightarrow V^{\mathbb{C}} = V \oplus iV$ (direct sum over $\mathbb{R}$)

• $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V^{\mathbb{C}}$

• $f: V \to W$, def. $f^{\mathbb{C}}: V^{\mathbb{C}} \to W^{\mathbb{C}}$ by $f(v_1, v_2) = (f(v_1), f(v_2))$

$\Rightarrow$ matrices of f and $f^{\mathbb{C}}$ the same $\Rightarrow$ eigenvalues the same

Application:

Thm.: Let $f \in \mathfrak{so}(V)$, with V a real vector space, $\dim V \geq 1$. Then f has an invariant subspace of dimension 1 or 2.

Proof: If f has real eigenvalue $\Rightarrow$ subspace spanned by some eigenvector invariant.

If not, all eigenvalues are complex.

$\Rightarrow$ choose $\lambda + i\mu \Rightarrow$ also eigenvalue of $f^{\mathbb{C}}$, let eigenvector be $v_1 + iv_2$ (in $V^{\mathbb{C}}$)

$\Rightarrow f^{\mathbb{C}}(v_1 + iv_2) = f(v_1) + i f(v_2) = (\lambda + i\mu)(v_1 + iv_2) = \underbrace{(\lambda v_1 - \mu v_2)}_{= f(v_1)} + i \underbrace{(\mu v_1 + \lambda v_2)}_{f(v_2)}$

linear span of v_1, v_2 in V invariant under f. $\square$

now:

$$V \to V^{\mathbb{C}} \to (V^{\mathbb{C}})_{\mathbb{R}} \cong V \oplus V \quad \text{canonically isomorphic}$$

L complex vector space: $L \to L_{\mathbb{R}} \to (L_{\mathbb{R}})^{\mathbb{C}}$?

<u>Def.:</u> The complex conjugate space $\overline{L}$ is def. as L with multiplication $\bar{c}\ell$ for $c \in \mathbb{C}, \ell \in L$.

then: $(L_{\mathbb{R}})^{\mathbb{C}} \underset{\text{can.}}{\cong} L \oplus \overline{L}$ (note: i on L, and J on $(L_{\mathbb{R}})^{\mathbb{C}}$)

Why?

$J^2 = -\mathrm{id}$, so J has eigenvalues $\pm i$

$$\Rightarrow \text{subspaces} \quad L^{1,0} = \{(\ell_1, \ell_2) \in (L_{\mathbb{R}})^{\mathbb{C}} : \overset{(-\ell_2, \ell_1)}{\overset{\parallel}{J(\ell_1, \ell_2)}} = i(\ell_1, \ell_2)\}$$

$$L^{0,1} = \{(\ell_1, \ell_2) \in (L_{\mathbb{R}})^{\mathbb{C}} : J(\ell_1, \ell_2) = -i(\ell_1, \ell_2)\}$$

$$\Rightarrow L^{1,0} \ni (\ell, -i\ell)$$

$$L^{0,1} \ni (m, im)$$