Session 16
Nov. 12, 2018

II. Inner Products and Geometry

II.1 Basics of Inner Products

Def.: Let L_1, L_2, M be vector spaces over the field $\mathbb{F}$. Then

$f: L_1 \times L_2 \to M$, $(l_1, l_2) \mapsto f(l_1, l_2)$ is called a bilinear map if

- $f(l_1 + l_1', l_2) = f(l_1, l_2) + f(l_1', l_2)$
  $f(l_1, l_2 + l_2') = f(l_1, l_2) + f(l_1, l_2')$ $\quad \forall l_1, l_1' \in L_1, l_2, l_2' \in L_2$
- $f(al_1, bl_2) = ab\, f(l_1, l_2)$ $\quad \forall l_1 \in L_1, l_2 \in L_2, a, b \in \mathbb{F}$

Def.: If $M = \mathbb{F}$ then such f are called bilinear forms.

Ex.:
- $\mathcal{L}(L, M) \times L \to M$, $(g, l) \mapsto g(l)$ bilinear map
- $L^* \times L \to \mathbb{F}$, $(h, l) \mapsto h(l)$ bilinear form
- $\mathbb{F}^n \times \mathbb{F}^m \to \mathbb{F}$, $(\vec{x}, \vec{y}) \mapsto \sum_{i,j} g_{ij} x_i y_j = \vec{x}^T G \vec{y}$, G $n \times m$ matrix
  bilinear form

Note: in gen. one can define multilinear mappings $L_1 \times L_2 \times \ldots \times L_n \to M$,
e.g., $L_i = \mathbb{R}^n$, $f(\vec{x}_1, \ldots, \vec{x}_n) = \det(\underbrace{\vec{x}_1 \ldots \vec{x}_n}_{\text{matrix with } \vec{x}_1, \ldots, \vec{x}_n \text{ as columns}})$

↳ leads to study of tensors ... later or next year.

We study bilinear forms $L \times L \to \mathbb{F}$ or $L \times \overline{L} \to \mathbb{C}$; these are called inner products.

Note: we write $L \times \overline{L} \to \mathbb{C}$ as sesquilinear form $g: L \times L \to \mathbb{C}$, i.e.,

$$g(a l_1, b l_2) = a \overline{b}\, g(l_1, l_2)$$ (linear in first, antilinear in second argument)

Next:
- inner products and matrices
- symmetry
- classification in 1 or 2 dim.
- gen. classification

Inner Products and Matrices

here:
- consider $L \times \overline{L} \to \mathbb{C}$, remove $\overline{}$ for $L \times L \to F$
- $\dim L = n < \infty$
- (L, g) an inner product space

choose basis $\{e_1, \dots, e_n\}$ in L, def. Gram matrix $G = \left(g(e_i, e_j)\right)_{i,j=1,\dots,n}$

this def. g: $\quad g(\vec{x}, \vec{y}) = g\left(\sum_{i=1}^{n} x_i e_i, \sum_{j=1}^{n} y_j e_j\right) = \sum_{i,j=1}^{n} x_i \overline{y_j}\, g(e_i, e_j) = \vec{x}^T G \overline{\vec{y}}$

also, choice of G and basis def. g

basis change: $\vec{x} = A \vec{x}\,' \implies g(\vec{x}, \vec{y}) = \vec{x}^T G \overline{\vec{y}} = (A\vec{x}\,')^T G\, (\overline{A \vec{y}\,'})$

$$= \vec{x}\,'\underbrace{A^T G \overline{A}}_{\text{Gram matrix in new basis}} \overline{\vec{y}\,'}$$

note: g can be def. via map $\tilde{g}: L \to \overline{L^*}$: $\tilde{g}(l)(m) =: g(l, m)$

fix basis $\{e_1, \dots, e_n\}$, g def. via matrix $G \implies$ matrix of $\tilde{g}$ in same basis and dual basis is G^T:

$$g(\vec{x},\vec{y}) = \tilde{g}(\vec{x})(\vec{y}) = \tilde{g}(\vec{x})^T\vec{\overline{y}} = (G^T\vec{x})^T\vec{\overline{y}} = \vec{x}^T G\vec{\overline{y}}$$

dual basis: $\vec{x}(\vec{y}) = \sum_{i=1}^{n} x_i e_i^* \sum_{j=1}^{n} y_j e_j = \sum_{i=1}^{n} x_i y_i = \vec{x}^T\vec{y}$

Symmetries:

bilinear: $g^T(l,m) := g(m,l)$; G changes to G^T

sesquilinear: $\overline{g}^T(l,m) := \overline{g(m,l)}$ (still linear in first argument); G changes to $\overline{G}^T$

We consider (from now on):

- symmetric: $g^T = g$ (G symmetric), orthogonal geometry
- antisymmetric or symplectic: $g^T = -g$ (G antisymmetric)
- Hermitian: $\overline{g}^T = g$ (G Hermitian)

(note: $\overline{g}^T(l,l) = \overline{g}(l,l) = \overline{g(l,l)} = g(l,l)$ is real)

Def.: Let (L,g) be an inner product space. Then l_1, l_2 are called orthogonal if $g(l_1,l_2) = 0$. $L_1, L_2 \subset L$ are orthogonal if $g(l_1,l_2) = 0 \ \forall l_1 \in L_1, l_2 \in L_2$.

note: if $g = \pm g^T$ then $g(l_1,l_2) = 0 \Leftrightarrow \pm g^T(l_1,l_2) = 0 \Leftrightarrow g(l_2,l_1) = 0$

Def.:
- $\ker g = \{m \in L : g(m,l) = 0 \ \forall l \in L\}$
- if $\ker g = \{0\}$ then g is called non-degenerate

note:

- $\ker g = \ker \tilde{g}$ ($\tilde{g}$ as above)
- g non-degenerate $\iff G$ non-singular
- $\operatorname{rank} g := \dim \operatorname{im} \tilde{g} = \operatorname{rank} G$