Session 17
Nov. 14, 2018

II.2 Classification for dim 1 or 2

Def.: Let (L_1, g_1) and (L_2, g_2) be inner product spaces.
A linear isomorphism $f: L_1 \to L_2$ is called isometry if
$$g_1(e, e') = g_2(f(e), f(e')) \quad \forall e, e' \in L_1.$$
(L_1, g_1) and (L_2, g_2) are called isometric if there is an isometry for them.

next: • classify spaces up to isometry
• want: $L = \bigoplus_{i=1}^{m} L_i$ with orthogonal L_i (dim $L_i = 1$ for orth. and Hermitian, dim $L_i = 1$ or 2 for symplectic)
$\Rightarrow$ study dim $L = 1$ or 2 first

Different cases:

1d orthogonal over $\mathbb{R}$:

a) $g = 0$ (degenerate)
↳ null

b) isometric to $g(x, y) = xy$ (x and y coordinates in some basis)
↳ positive (since $g(x,x) > 0$ for $x \neq 0$) (non-degenerate)

c) isometric to $g(x, y) = -xy$ (non-degenerate)
↳ negative

Proof: choose any $e \in L$, if $g(e,e)=0 \Rightarrow g(xe,e)=0 \Rightarrow g=0$

otherwise $g(e,e) \neq 0$, call $g(e,e)=a$, so $g(xe,ye)=axy$

basis change $e \mapsto \frac{1}{\sqrt{|a|}} e$

$$\Rightarrow g\left(x\frac{e}{\sqrt{|a|}}, y\frac{e}{\sqrt{|a|}}\right) = \frac{a}{|a|}xy = \begin{cases} xy, & a>0 \\ -xy, & a<0 \end{cases}$$

1d symmetric over $\mathbb{C}$:

a) $g=0$

b) $g(x,y)=xy$

Proof: here we can do basis change $e \mapsto \frac{1}{\sqrt{a}} e$

1d Hermitian (over $\mathbb{C}$):

a) $g=0$

b) $g(x,y)=x\bar{y}$

c) $g(x,y)=-x\bar{y}$

Proof: if $g(e,e)=0 \Rightarrow g=0$ as before

otherwise $0 \overset{(x\neq 0)}{\neq} g(xe,xe) = |x|^2 \underbrace{g(e,e)}_{=e^{i\alpha}}$ classifies g

but $g(e,e)=\overline{g(e,e)}$, so $g(e,e)=\pm 1$

1d symplectic (over $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ (in gen.: characteristic $\neq 2$)):

a) $g = 0$

Proof: $g(e,e) = -g(e,e) \Rightarrow 2g(e,e) = 0 \Rightarrow g(e,e) = 0$

$$g(xe, ye) = xy\, g(e,e) = 0$$

2d symplectic:

a) $g = 0$

b) $g(x_1 e_1 + x_2 e_2, y_1 e_1 + y_2 e_2) = x_1 y_2 - x_2 y_1$ in some basis $\{e_1, e_2\}$

note: Gram matrix $G = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Proof: • g degenerate

$\hookrightarrow \exists\, e \neq 0$ s.t. $g(e,m) = 0 \;\forall m \in L \quad (*)$

extend to basis $\{e, e'\}$

$\Rightarrow g(x_1 e + x_2 e', y_1 e + y_2 e')$

$$= x_1 y_1 \underbrace{g(e,e)}_{=0 \text{ as before}} + x_1 y_2 \underbrace{g(e,e')}_{=0\ (*)} + x_2 y_1 \underbrace{g(e',e)}_{=-g(e,e') = 0\ (*)} + x_2 y_2 \underbrace{g(e',e')}_{=0}$$

• g non-degenerate

$\hookrightarrow \exists\, \tilde{e}_1, e_2$ s.t. $g(\tilde{e}_1, e_2) = a \neq 0$, or, for $e_1 = \frac{\tilde{e}_1}{a}$, $g(e_1, e_2) = 1$

suppose $e_1 = c e_2 \Rightarrow g(e_1, e_2) = g(c e_2, e_2) = c\, g(e_2, e_2) = 0$,

so e_1 and e_2 are linearly independent

$\Rightarrow$ as above: $g(x_1 e_1 + x_2 e_2, y_1 e_1 + y_2 e_2)$

$$= x_1 y_1 \underbrace{g(e_1, e_1)}_{=0} + x_1 y_2 \underbrace{g(e_1, e_2)}_{=1} + x_2 y_1 \underbrace{g(e_2, e_1)}_{=-g(e_1,e_2)=-1} + x_2 y_2 \underbrace{g(e_2, e_2)}_{=0}$$

II.3 General Classification

Def.: A subspace $L_0 \subset L$ is called

- non-degenerate if $g|_{L_0}$ is non-degenerate
- isotropic if $g|_{L_0} = 0$

Ex.: $\left(\mathbb{R}^2, g\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = x_1 y_1 - x_2 y_2\right)$

$\hookrightarrow$ $\operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ non-degenerate

$\hookrightarrow$ $\operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ isotropic

Def.: The orthogonal complement $L_0^\perp$ of $L_0 \subset L$ is

$$L_0^\perp := \{ e \in L : g(e_0, e) = 0 \text{ for all } e_0 \in L_0 \} \subset L$$