Session 18
Nov. 19, 2018

Lemma: Let (L, g) be an inner product space with $\dim L < \infty$. Then

a) $L_0 \subset L$ non-degenerate $\Rightarrow L = L_0 \oplus L_0^\perp$
(in particular: $\dim L = \dim L_0 + \dim L_0^\perp$)

b) $L_0 \subset L$ and $L_0^\perp$ non-degenerate $\Rightarrow (L_0^\perp)^\perp = L_0$

Proof: a) as before, def. $\tilde{g}: L \to L^*$, s.t. $\tilde{g}(\ell)(m) = g(\ell, m)$

def. $\tilde{g}|_{L_0}: L_0 \to L^*$, L_0 non-degenerate $\Rightarrow \ker \tilde{g}|_{L_0} = 0$

$\Rightarrow \dim \operatorname{im} \tilde{g}|_{L_0} = \dim L_0$

$\Rightarrow L^* \supset g(L_0, \cdot)$ has dimension $\dim L_0$

$\Rightarrow$ choose basis for $g(L_0, \cdot)$, extend to basis of L^*, $\dim L^* = \dim L$

$\Rightarrow \dim L_0^\perp = \dim L - \dim L_0$ and $L_0 + L_0^\perp$ is a direct sum

b) $L_0 \subset (L_0^\perp)^\perp$, if both non-degenerate, then

$\dim (L_0^\perp)^\perp = \dim L - \dim L_0^\perp = \dim L_0$ $\square$

This lemma gives us the desired decomposition of L:

Thm.: Let (L,g) be an inner product space with $\dim L < \infty$. Then $L = \bigoplus_{i=1}^{m} L_i$,

where the L_i's are pairwise orthogonal and

- 1-dimensional for symmetric and Hermitian forms,
- 1-dimensional degenerate or 2-dimensional non-degenerate for symplectic forms.

Proof: Induction in $\dim L$.

$\dim L = 1$ clear, so let $\dim L \geq 2$. If $g = 0$ clear, so let $g \neq 0$.
Induction hypothesis: For $1, 2, \ldots, \dim L - 1$ dimensional spaces we have the desired decomposition. Now consider $\dim L$ dimensional space L.

- symplectic case:

$\exists\, e_1, e_2$ s.t. $g(e_1, e_2) \neq 0$, actually $L_0 = \operatorname{span}\{e_1, e_2\}$ non-degenerate (as in II.2)

$\Rightarrow$ $L = L_0 \oplus L_0^{\perp}$, use induction hypothesis for $L_0^{\perp}$

- symmetric case:

assume $g(e,e) = 0$ for all $e \in L$. Then for any $e_1, e_2 \in L$:

$$0 = g(e_1 + e_2, e_1 + e_2) = \underbrace{g(e_1, e_1)}_{=0} + 2g(e_1, e_2) + \underbrace{g(e_2, e_2)}_{=0} = 2g(e_1, e_2)$$

$\Rightarrow g(e_1, e_2) = 0 \Rightarrow g = 0$

so $g(e_0, e_0) \neq 0$ for some $e_0 \in L$

$\Rightarrow$ take $L_0 = \operatorname{span}\{e_0\}$, then use $L = L_0 \oplus L_0^{\perp}$ and induction hypothesis

- Hermitian case:

as in symm. case, let $g(\ell,\ell)=0\ \forall \ell\in L$, then $\forall\, \ell_1,\ell_2\in L$:

$$0=g(\ell_1+\ell_2,\ell_1+\ell_2)=\underbrace{g(\ell_1,\ell_1)}_{=0}+g(\ell_1,\ell_2)+\underbrace{g(\ell_2,\ell_1)}_{=\overline{g(\ell_1,\ell_2)}}+\underbrace{g(\ell_2,\ell_2)}_{=0}=2\,\mathrm{Re}\, g(\ell_1,\ell_2)$$

$\Rightarrow g(\ell_1,\ell_2)=ia$ for some $a\in\mathbb{R}$, say $a\neq 0$

$\Rightarrow\ 0=\mathrm{Re}\, g\left(\frac{\ell_1}{ia},\ell_2\right)=\mathrm{Re}\, \frac{1}{ia}\, g(\ell_1,\ell_2)=\mathrm{Re}\, 1=1$ is a contradiction

$\Rightarrow a=0$, so $g=0$ and conclude as in symm. case $\square$

Can these inner products be classified uniquely, as discussed in II.2, up to isometry? Yes, as we will prove next.

Classification according to the following invariants:

- $n=\dim L$, $r_0=\dim \ker g$
- for symm. and Hermitian: given $L=\bigoplus_{i=1}^{n} L_i$ as in Thm. then
  $r_+=$ number of positive L_i (as in II.2, $g(x,x)>0$)
  $r_-=$ number of negative L_i ($g(x,x)<0$)

$\Rightarrow n=r_0+r_++r_-$ and (r_0,r_+,r_-) is called signature of (L,g)

(sometimes it's called inertia, and r_+-r_- is called signature (if $r_0=0$))