Session 21
Nov. 28, 2018

II.5 Euclidean and Unitary Spaces

<u>Def.:</u> A Euclidean space (L,g) is a real vector space L, $\dim L<\infty$, with symm. and positive definite inner product g ($g(l,l)>0$ for $l\neq 0$, or $r_0=r_-=0$)

A unitary space (L,g) is a complex vector space L with Hermitian and pos. def. inner product g.

notation: $g(l,m)=:\langle l,m\rangle$, $\sqrt{\underbrace{\langle l,l\rangle}_{>0}}=:\|l\|$ (length of L)

usual terminology: such g are called scalar products

Remarks:

- $\dim L<\infty$: both spaces have orthonormal basis
  $\Rightarrow$ isometric to $\mathbb{R}^n$, $\mathbb{C}^n$ with canonical scalar product
  $$\langle\vec{x},\vec{y}\rangle=\sum_{i=1}^n x_i\overset{(-)}{y_i}, \quad \|\vec{x}\|=\sqrt{\sum_{i=1}^n |x_i|^2}$$
- Cauchy-Schwarz (-Bunyakowskii): $|\langle l_1,l_2\rangle|\le\|l_1\|\cdot\|l_2\|$ with equality iff l_1,l_2 linearly dependent (proof: see Adv. Calc.; use $0\le\langle\lambda l_1+\mu l_2,\lambda l_1+\mu l_2\rangle$ and choose λ,μ right)
- C-S implies $\|l_1+l_2\|^2=\|l_1\|^2+\langle l_1,l_2\rangle+\langle l_2,l_1\rangle+\|l_2\|^2$
  $\le\|l_1\|^2+2\|l_1\|\cdot\|l_2\|+\|l_2\|^2=(\|l_1\|+\|l_2\|)^2$

i.e., the triangle inequality: $\|l_1+l_2\|\le\|l_1\|+\|l_2\|$ holds

- thus $\|\cdot\|$ is indeed a norm and $d(e_1,e_2)=\|e_1-e_2\|$ is a metric

recall: norm $\|\cdot\|$: • $\|e\|=0 \Leftrightarrow e=0$ (positive definite)

• $\|\lambda e\| = |\lambda|\cdot\|e\|$ (absolutely homogeneous)

• $\|e_1+e_2\| \leq \|e_1\|+\|e_2\|$ (triangle inequality)

metric $d(\cdot,\cdot)$: • $d(e_1,e_2)\geq 0$ (non-negative)

• $d(e_1,e_2)=0 \Leftrightarrow e_1=e_2$

• $d(e_1,e_2)=d(e_2,e_1)$ (symmetry)

• $d(e_1,e_3)\leq d(e_1,e_2)+d(e_2,e_3)$ (triangle inequality)

- unitary spaces complete w.r.t. $\|e_1-e_2\|=\sqrt{\langle e_1-e_2,e_1-e_2\rangle}$ are called Hilbert spaces

  ↳ for $\dim L<\infty$ all unitary spaces are Hilbert spaces

(any complete normed space (norm is not necessarily def. via scalar product, see HW) is called Banach space)

- angles (Euclidean space): due to C-S: $-1 \leq \frac{\langle e_1,e_2\rangle}{\|e_1\|\cdot\|e_2\|} \leq 1$

$\Rightarrow \exists\, \varphi\in[0,\pi]$ s.t. $\cos\varphi = \frac{\langle e_1,e_2\rangle}{\|e_1\|\cdot\|e_2\|}$, $\varphi=\measuredangle(e_1,e_2)=$ angle between e_1 and e_2

Is this really an angle in $\mathbb{R}^2$ or any plane?

consider $e_1'=\frac{e_1}{\|e_1\|}=\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}$, $e_2'=\frac{e_2}{\|e_2\|}=\begin{pmatrix}\cos\beta\\ \sin\beta\end{pmatrix}$

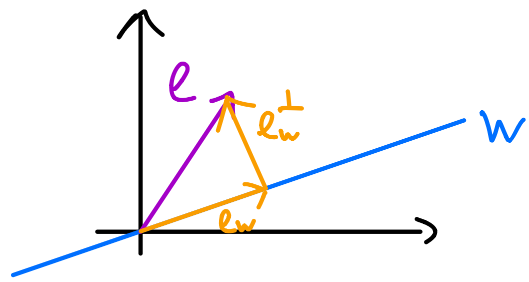
$\Rightarrow \langle e_1',e_2'\rangle = \cos\alpha\cos\beta+\sin\alpha\sin\beta=\cos(\beta-\alpha)$ ✓

- distance (Euclidean space) between $V, W \subset L$ is def. as

$$d(U,W) = \min\{\|e_1 - e_2\| : e_1 \in V, e_2 \in W\}$$

let $V = \{e\}$, $W \subset L$ a subspace (a hyperplane through the origin);
we know $L = W \oplus W^\perp$, so $e = \underbrace{e_w}_{\in W} + \underbrace{e_w^\perp}_{\in W^\perp}$ uniquely ($e_w, e_w^\perp$ called orthogonal projections)

Claim: $d(\{e\}, W) = \|e_w^\perp\|$



Proof: for any $w \in W$:

$$\|e-w\|^2 = \|e_w + e_w^\perp - w\|^2 = \langle e_w - w + e_w^\perp, e_w - w + e_w^\perp\rangle$$

$$= \|e_w - w\|^2 + \|e_w^\perp\|^2 \geq \|e_w^\perp\|^2$$

$e_w^\perp \perp \underbrace{e_w - w}_{\in W}$

$\Rightarrow$ minimum when $\|e-w\| = \|e_w^\perp\|$, i.e., $w = e_w$. $\square$

explicit formula: given basis $\{e_1, \dots, e_m\}$ of W: $e_w = \sum_{i=1}^m \langle e, e_i\rangle e_i$,

so $d(\{e\}, W) = \|e - e_w\| = \|e - \sum_{i=1}^m \langle e, e_i\rangle e_i\|$

Pythagoras: $\|e_w\|^2 = \|e\|^2 - \|e_w^\perp\|^2 \leq \|e\|^2$, so $\sum_{i=1}^m \langle e, e_i\rangle^2 \leq \|e\|^2$

note: in ∞-dim. Hilbert space: $\sum_{i=1}^\infty |\langle e, e_i\rangle|^2 \leq \|e\|^2$ (Bessel inequality)
which will give us convergence of $\sum_{i=1}^\infty \langle e, e_i\rangle e_i$ (to e if $\{e_i\}$ is an orthonormal Hilbert-space basis)

- volume: need additive, monotone, multiplicative for orth. direct sums;
also: we will see later that any linear $f: L \to L$ can be written as

$f = U \cdot \tilde{f}$, so f stretches each direction with

U: isometry; $\tilde{f}$: diagonalizable

$|\lambda_1| \cdot \ldots \cdot |\lambda_n| = |\det \tilde{f}|$

λ_i: eigenvalues

$\Rightarrow \text{volume}(f(U)) = |\det f| \cdot \text{volume}(U)$

<u>Ex.</u>: parallelepiped with sides $\{v_1, \ldots, v_n\}$

$\{v_1, \ldots, v_n\}$ lin. dep. $\Rightarrow$ vol $= 0$

lin. indep.: consider basis $\{e_1, \ldots, e_n\}$ and f mapping e_i to v_i, i.e., its matrix is A with $(v_1, \ldots, v_n) = (e_1, \ldots, e_n) A$

$\Rightarrow G(\{v_i\}) := (\langle v_i, v_j \rangle)_{i,j=1,\ldots,n} = A^T A$ (Gram matrix of $\{v_i\}$)

$$\begin{aligned}\Rightarrow \text{volume} = \text{volume}(f(\text{unit cube})) &= |\det f| \cdot \text{volume}(\text{unit cube}) \\ &= |\det A| \\ &= \sqrt{\det G}\end{aligned}$$