Session 22
Dec. 3, 2018

- unitary spaces and decomplexification:

↳ norm: $\{e_j\}_{j=1,\dots,n}$ ONB of L, $\{e_j, ie_j\}_{j=1,\dots,n}$ ONB of $L_{\mathbb{R}}$ (the decomplexification of L)

$$\Rightarrow \underbrace{\left\| \sum_{j=1}^{n} x_j e_j \right\|^2}_{\text{norm on } L} = \sum_{j=1}^{n} |x_j|^2 = \sum_{j=1}^{n} \left[(\mathrm{Re}\, x_j)^2 + (\mathrm{Im}\, x_j)^2 \right]$$

$$= \underbrace{\left\| \sum_{j=1}^{n} (\mathrm{Re}\, x_j) e_j + \sum_{j=1}^{n} (\mathrm{Im}\, x_j)(ie_j) \right\|^2}_{\text{norm on } L_{\mathbb{R}}}$$

↳ scalar product: set $\langle l, m \rangle = \underbrace{\mathrm{Re}\langle l, m\rangle}_{a(l,m)} + i \underbrace{\mathrm{Im}\langle l, m\rangle}_{b(l,m)}$

from $\langle l, m \rangle = \overline{\langle m, l \rangle}$ we get
- $a(l,m) = a(m,l)$ (symm.)
- $b(l,m) = -b(m,l)$ (antisymm.)

$\Rightarrow$ $\langle \cdot, \cdot \rangle$ pos. def. $\Leftrightarrow$ $a(\cdot,\cdot)$ pos. def.

further:
- $a(il, im) = a(l,m)$
- $b(il, im) = b(l,m)$
- $a(l,m) = b(il, m)$
- $b(l,m) = -a(il, m)$

results:
- $\{e_j\}_{j=1,\dots,n}$ basis of L, then $\{e_1, \dots, e_n, ie_1, \dots, ie_n\}$ ONB for a, and symplectic basis for b
- other way around: complex structure for any $2n$-dim. Euclidean space is $J(e_j) = e_{j+n}$, $J(e_{j+n}) = -e_j$, $j = 1, \dots, n$

II.6 Orthogonal and Unitary Operators

L finite dim. Euclidean (unitary) space with scalar product $\langle\cdot,\cdot\rangle$

recall: $f: L \to L$ isomorphism with $\langle f(l), f(m)\rangle = \langle l, m\rangle \iff f$ isometry

isometries on Euclidean (unitary) spaces are called orthogonal (unitary) operators

<u>Lemma:</u> f isometry if and only if

a) $\|f(l)\| = \|l\| \ \forall l \in L$

b) $\{e_j\}_{j=1,\dots,n}$ basis of L, G Gram matrix of $\langle\cdot,\cdot\rangle$, U matrix of f, then

$$U^T G \overset{(-)}{U} = G$$

c) f maps any ONB into another ONB

d) matrix U of f in any ONB satisfies $U^T \overset{(-)}{U} = E_n$, i.e., $U^{-1} = \overset{(-)}{U}{}^T$

<u>Proof:</u> a) clear from polarization (for $\mathrm{Re}\langle l, m\rangle$ in Hermitian case)

b) clear by def. of Gram matrix

c) clearly $\iff \langle f(l), f(m)\rangle = \langle l, m\rangle \ \forall l, m$

d) clearly $\iff$ c)

note: • group $O(n) = \{$orth. $n\times n$ matrices$\}$

• group $U(n) = \{$unitary $n\times n$ matrices$\}$

• $|\det U|^2 = \det U \cdot \overline{\det U} = \det U \cdot \det \overset{(-)}{U}{}^T = \det U \overset{(-)}{U}{}^T = \det E_n = 1$

- $SO(n) = O(n) \cap \{\det = +1\}$

Low dimension:

- $n=1$: $U(1) = \{e^{ix} : x \in \mathbb{R}\}$, $O(1) = \{\pm 1\}$
- $n=2$: consider $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(2)$

↳ $\det U = +1$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = 1$

$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ for some $\varphi \in [0, 2\pi)$

$\Rightarrow$ rotation by angle φ

note: not diagonalizable unless $\varphi \in \{0, \pi\}$

↳ $\det U = -1$: $U = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ -\sin\varphi & -\cos\varphi \end{pmatrix}$ diagonalizable with eigenvalues ± 1 and orthogonal eigenspaces

$\Rightarrow$ reflection relative to some line

Thm.: a) f unitary $\Longleftrightarrow$ f diagonalizable in some ONB with $|\lambda_j| = 1$ (λ_j eigenvalue, $j = 1, \dots, n$)

b) f orthogonal $\Longleftrightarrow$ in some ONB the matrix of f is

$$U = \begin{pmatrix} U(\varphi_1) & & & & & & 0 \\ & \ddots & & & & & \\ & & U(\varphi_n) & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & 1 & & \\ & & & & & -1 & \\ 0 & & & & & & \ddots \\ & & & & & & -1 \end{pmatrix}, \text{ with } U(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \; \varphi \notin \{0, \pi\}$$

note: eigenvectors to different eigenvalues are orthogonal: gen. argument:

$$f(e_i) = \lambda_i e_i \Rightarrow \langle e_1, e_2 \rangle = \langle f(e_1), f(e_2) \rangle = \lambda_1 \overline{\lambda_2} \langle e_1, e_2 \rangle$$

if $\lambda_1 \neq \lambda_2$ then $\lambda_1 \overline{\lambda_2} \neq 1$ $(|\lambda_i| = 1)$ so $\langle e_1, e_2 \rangle = 0$

Corollary (Soccer thm.):

There are two points on the surface of a soccer ball, which at the beginning of the first and the second half are at the same point in space.

(Euler: $SO(3)$ = rotation around some axis)

Proof: rotation in 3-dim.: $\det U = +1$

$\Rightarrow$ there is an eigenvalue $+1$

Proof of thm.:

a) "$\Leftarrow$" clear

"$\Rightarrow$" λ eigenvalue with characteristic 1-dim. subspace L_λ $(f(L_\lambda) \subset L_\lambda)$; since f unitary $\lambda \in U(1)$, i.e., $|\lambda| = 1$

from II.3: $L = L_\lambda \oplus L_\lambda^\perp$ and $\langle \underbrace{e_\lambda}_{\in e_\lambda}, \underbrace{f(e_\lambda^\perp)}_{\in L_\lambda^\perp} \rangle = \langle f(\lambda^{-1} e_\lambda), f(e_\lambda^\perp) \rangle$

$$= \lambda^{-1} \langle e_\lambda, e_\lambda^\perp \rangle = 0$$

so also $L_\lambda^\perp$ f-invariant

$\Rightarrow$ induction in $\dim L$ proves statement

b) "$\Leftarrow$" clear

"$\Rightarrow$" if f has real eigenvalue, proceed as before

otherwise: use thm. from I.10 that real vector space has 1 or 2 dim. invariant subspace $\square$