Jacobs University
Fall 2018

December 12, 2018

Linear Algebra

Final Exam

Instructions:

- Do all the work on this exam paper.
- Show your work, i.e., carefully write down the steps of your solution.
- Calculators and other electronic devices and notes are not allowed.
- You are free to refer to any results proven in class or the homework sheets unless stated otherwise (and unless the problem is to reproduce a result from class or the homework sheets), but you need to explicitly mention which one you refer to.

Name: ______________________________

Problem 1: Diagonalization [20 points]

Let L be a finite dimensional vector space, and let $f : L \to L$ be a linear map.

(a) Define the following notions: *invariant subspace*, *eigenvalue*, *eigenvector*. Also define what it means for f to be diagonalizable (without referring to matrices).

(b) Define what the characteristic polynomial P_f is and prove that λ is an eigenvalue of f if and only if λ is a root of P_f.

(c) Define what the spectrum of f is and what it means for the spectrum to be simple.

(d) Assume f is diagonalizable with simple spectrum. Prove that any linear map $g : L \to L$ with the property that $fg - gf = 0$ can be written as a polynomial of f.

Problem 1: Extra Space

Problem 1: Extra Space

Problem 1: Extra Space

Problem 2: Jordan Decomposition [20 points]

Let L be a finite dimensional vector space over some algebraically closed field, and let $f : L \to L$ be a linear map.

(a) Define what it means for a linear map f to be nilpotent.

(b) Define what a generalized eigenvector and a generalized eigenspace are.

(c) In class we discussed the abstract Jordan decomposition. State what this is (without referring to matrices). Is this theorem still true if the field is not algebraically closed (just state your answer and give a very brief reason here)?

(d) Let A be a complex 2×2 matrix. Suppose A has only one eigenvalue λ. Does this imply that A is diagonalizable? Give either a proof or a counter example.

(e) Let $J_r(\lambda)$ be a complex $r \times r$ matrix with only eigenvalue λ. Define what is means for $J_r(\lambda)$ to be a Jordan block. Now assume $J_r(\lambda)$ is a Jordan block. Compute and explicitly write down the matrix $J_4(\lambda)^n$.

Problem 2: Extra Space

Problem 2: Extra Space

Problem 2: Extra Space

Problem 3: Bilinear Forms [20 points]

Let L be a finite dimensional vector space over a field F.

(a) Define what an inner product on L is. Also define what symmetric, symplectic and Hermitian inner products are.

(b) Let g be an inner product on L, and $\{e_1, \ldots, e_n\}$ a basis of L. Define what the Gram matrix of g is.

(c) Let g be an inner product on L. Define what orthogonality of two vectors $\ell_1, \ell_2 \in L$ means, what $\ker g$ is, and what it means for g to be non-degenerate. Then suppose that g is non-degenerate and that $\dim L = n$. Prove that if the matrix $(g(e_i, e_j))_{i,j=1,\ldots,n}$ is non-singular then the vectors $e_1, \ldots, e_n \in L$ are linearly independent.

(d) Let (L, g) be a *Hermitian* inner product space. Define what an isometry is. Then state how we classified g up to isometry in class in the one-dimensional case.

Problem 3: Extra Space

Problem 3: Extra Space

Problem 3: Extra Space

Problem 4: Euclidean and Unitary Spaces [20 points]

Let L be a finite dimensional vector space.

(a) Define what a Euclidean and what a unitary space are. What is the natural metric on such spaces?

(b) On a Euclidean space, let a norm $\|\cdot\|$ be defined such that $\|\ell\| = \sqrt{\langle \ell, \ell \rangle}$ ($\langle \cdot, \cdot \rangle$ denotes a scalar product) for all $\ell \in L$. Prove that this norm satisfies the identity

$$\|\ell_1 + \ell_2\|^2 + \|\ell_1 - \ell_2\|^2 = 2\|\ell_1\|^2 + 2\|\ell_2\|^2.$$

(c) For the Euclidean space $L = \mathbb{R}^2$, prove that there exists a norm $\|\cdot\|$ that is not given via any scalar product (i.e., via the relation $\|\ell\| = \sqrt{\langle \ell, \ell \rangle}$) by providing a counter example. Clearly prove the desired properties for your counter example.

Problem 4: Extra Space

Problem 4: Extra Space

Problem 4: Extra Space

Problem 5: Orthogonal, Unitary, and Self-adjoint Operators [20 points]
Let L be a finite dimensional Euclidean or unitary space.

(a) Define what orthogonal and unitary operators are.

(b) Prove that $|\det U| = 1$ for orthogonal and unitary operators.

(c) Are all unitary operators diagonalizable? Are all orthogonal operators diagonalizable? Here it is enough to just state the answer and the corresponding theorem we discussed in class.

(d) Define what a self-adjoint operator is.

(e) Suppose a linear map $f : L \to L$ is diagonalizable in some orthonormal basis with real eigenvectors. Prove that then f must be self-adjoint.

Problem 5: Extra Space

Problem 5: Extra Space

Problem 5: Extra Space

Bonus Problem [15 points]

Let L be an n-dimensional unitary space, and $f : L \to L$ linear. The adjoint of f is denoted by f^*. Then f^*f is a positive ($\langle \ell, f^*f\ell\rangle > 0$ for all $0 \neq \ell \in L$) self-adjoint operator, so it is diagonalizable with positive eigenvalues $s_1, \dots, s_n$ (if some eigenvalues are the same they are counted here with their multiplicities). So in particular one can define the square root of f^*f. The polar decomposition states that there is a unitary operator U such that $f = U\sqrt{f^*f}$. Given this, prove the following *singular value decomposition*: There exist two orthonormal bases $\{e_1, \dots, e_n\}$ and $\{\widetilde{e}_1, \dots, \widetilde{e}_n\}$ of L such that for all $\ell \in L$,

$$f(\ell) = \sum_{j=1}^{n} s_j \langle \ell, e_j\rangle \widetilde{e}_j.$$

(For the physicists: In bra-ket notation this can be nicely written as $f = \sum_j s_j |\widetilde{e}_j\rangle\langle e_j|$ and remains true for compact operators.)

Bonus Problem: Extra Space

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Bonus Problem: Extra Space