Jacobs University
Fall 2018

October 24, 2018

Linear Algebra

Midterm Exam

Instructions:

- Do all the work on this exam paper.
- Show your work, i.e., carefully write down the steps of your solution.
- Calculators and other electronic devices and notes are not allowed.
- You are free to refer to any results proven in class or the homework sheets unless stated otherwise (and unless the problem is to reproduce a result from class or the homework sheets), but you need to explicitly mention which one you refer to.

Name: ______________________________

Problem 1: Vector Space [25 points]
Recall that a vector space V over a field F is a set with addition and scalar multiplication which is associative (for addition and scalar multiplication), commutative (for addition), distributive (for scalars and vectors), and where an additive zero and inverse, and a multiplicative identity exist.

(a) Prove that the zero, and the inverse element to each $v \in V$ are unique.

(b) Consider the vector space of all real polynomials of degree n or smaller. What is the zero element? What is the inverse element for any given polynomial? What is a basis for this vector space? What is its dimension? For $n > 2$, is the space of all real polynomials of degree smaller or equal 2 a subspace (give a short explanation/proof)?

(c) Let $E \subset V$ and suppose that every $v \in V$ can be uniquely written as $v = \sum_{i=1}^{k} c_i e_i$ for some $k \in \mathbb{N}$, and $c_i \in F$, $e_i \in E$ for all $i = 1, \ldots, k$. Prove that then E is a minimal generating set.

Problem 1: Extra Space

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Problem 2: Linear Maps [25 points]
Let V, W be a vector spaces over the field F. Recall that $\mathcal{L}(V, W)$ denotes the set of all linear maps $V \to W$.

(a) Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(cv) = cf(v)$ for all $c \in \mathbb{R}$, $v \in \mathbb{R}^2$, but not $f(v_1 + v_2) = f(v_1) + f(v_2)$ for all $v_1, v_2 \in \mathbb{R}^2$.

(b) Give an example of a function $f : \mathbb{C} \to \mathbb{C}$ such that $f(z_1 + z_2) = f(z_1) + f(z_2)$ for all $z_1, z_2 \in \mathbb{C}$, but not $f(cz) = cf(z)$ for all $c, z \in \mathbb{C}$.

(c) Let E be a basis of V (which can be finite or countably infinite dimensional). Define what V^* (the dual space of V) and E^* (the dual basis) are. Prove that the dual basis is a linear independent set. Prove that for $\dim V < \infty$ the dual basis is indeed a basis, but that for countably infinite dimensional V this does not need to be true.

Problem 2: Extra Space

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Problem 3: (Direct) Sums [25 points]
Let V_1, V_2, W be non-empty subspaces of a finite dimensional vector space V over the field F.

(a) Suppose $V_1 + V_2$ is a direct sum. Prove that then $\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2)$.

(b) Either prove or give a counterexample to the assertion that $V = V_1 \oplus W$ and $V = V_2 \oplus W$ implies $V_1 = V_2$.

(c) Let $p_1, \ldots, p_n$ be projectors $V \to V$ with $\sum_{i=1}^{n} p_i = \mathrm{id}$ and $p_i p_j = 0$ for all $i \neq j$. Prove that then

$$V = \bigoplus_{i=1}^{n} \mathrm{im}(p_i).$$

Problem 3: Extra Space

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Problem 4: Quotient Spaces [25 points]
Let M be a subspace of the finite dimensional vector space L over the field F. Recall that we proved in class that $L^*/M^\perp$ is canonically isomorphic to M^*, and that $(L/M)^*$ is canonically isomorphic to $M^\perp$.

(a) Define what the quotient space L/M, the quotient map $q : L \to L/M$ and the orthogonal complement $M^\perp$ are.

(b) Prove that $\dim M + \dim M^\perp = \dim L$.

(c) Let f be a linear map between two finite dimensional vector spaces V, W over the field F. Define what the dual map f^* is. Prove that $\ker(f^*)$ is canonically isomorphic to $(\mathrm{coker}(f))^*$, and that $\mathrm{im}(f^*)$ is canonically isomorphic to $(\mathrm{coim}(f))^*$.

Problem 4: Extra Space

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