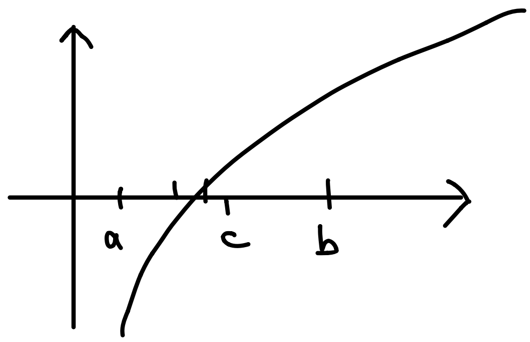
To calculate IRR, we need to find roots of $PV(y) - P = 0$

Session 3
Sep. 13, 2018

Methods:

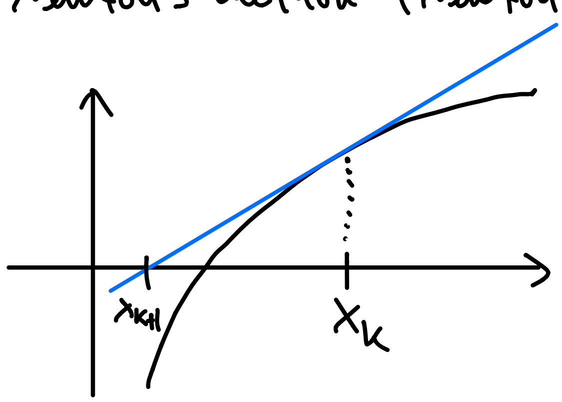
- Bisection:



- choose $a < b$, s.t. $f(a) \cdot f(b) < 0$
  (if $f(a)f(b) = 0$, we are done)
- set $c = \frac{a+b}{2}$
  - $\rightarrow$ if $f(c) = 0$, we're done
  - $\rightarrow$ if $f(a) \cdot f(c) < 0 \Rightarrow$ root is in $[a, c]$
  - $\rightarrow$ if $f(b) \cdot f(c) < 0 \Rightarrow$ root is in $[c, b]$

- repeat with either $[a, c]$ or $[c, b]$
- Advantages: • robust, only continuity necessary
  (except if $f(x) \geq 0 \ \forall x$)
- Disadvantage: • slow, linear convergence (error reduces by $\frac{1}{2}$ in each step)

- Newton's method (Newton-Raphson):



- we have: $f'(x_k) = \dfrac{f(x_k)}{x_k - x_{k+1}}$

$$\Rightarrow x_k - x_{k+1} = \frac{f(x_k)}{f'(x_k)}$$

$$\Rightarrow x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \longrightarrow \text{iterate}$$

- convergence?

use Taylor expansion around x_k

$$f(z) = f(x_k) + f'(x_k)(z - x_k) + \frac{f''(x_k)}{2}(z - x_k)^2 + \underbrace{O((z-x_k)^3)}_{=R}$$

Let z be the root, i.e., $f(z) = 0$

$$\Rightarrow \quad 0 = f(x_k) + f'(x_k)(z - x_k) + \frac{f''(x_k)}{2}(z - x_k)^2 + R$$

$$x_k = x_{k+1} + \frac{f(x_k)}{f'(x_k)}$$

$$\Rightarrow \quad 0 = f(x_k) + f'(x_k)\left(z - x_{k+1} - \frac{f(x_k)}{f'(x_k)}\right) + \frac{f''(x_k)}{2}(z - x_k)^2 + R$$

$$\Rightarrow \quad z - x_{k+1} = \frac{-f''(x_k)}{2f'(x_k)}(z - x_k)^2 + O((z - x_k)^3)$$

neglect

error in k-th step $\varepsilon_k = |z - x_k|$

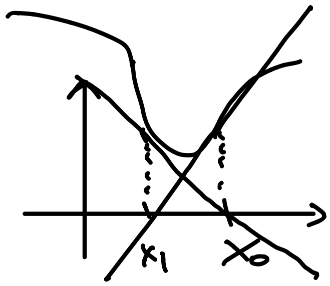
$$\Rightarrow \quad \varepsilon_{k+1} \leq \underbrace{\left|\frac{f''(x_k)}{2f'(x_k)}\right|}_{\leq C} \varepsilon_k^{2}$$

order of convergence

$\Rightarrow$ Newton's method converges quadratically!

- Advantages: • fast, quad. conv.
- Disadvantages: • need differentiability
  • need derivative explicitly

• may not converge: problems $\rightarrow$ $f'(x_k) = 0$ for some k

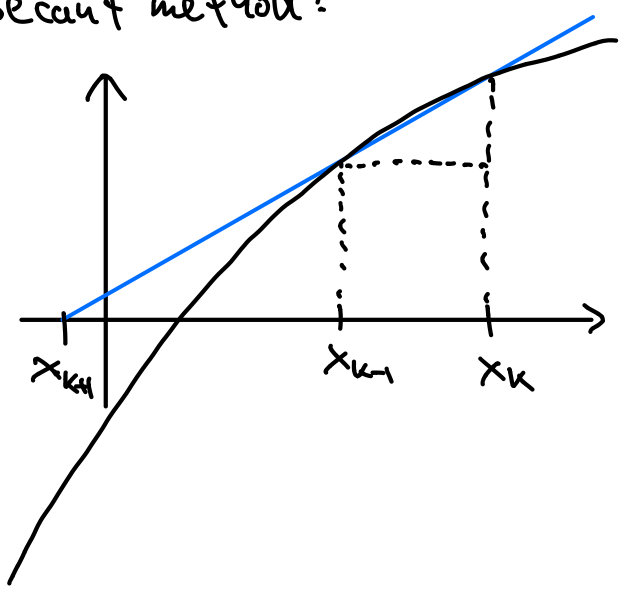


cyclic behavior

$\rightarrow$ f'' not cont.

$\rightarrow$ x_0 might be too far away from root

• Secant method:



– take secants instead of tangents:

Thales: $$\frac{f(x_k)}{x_k - x_{k+1}} = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

$$\Rightarrow x_k - x_{k+1} = \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

$$\Rightarrow x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

– order of convergence ~ 1.62 (Golden Ratio)

– Adv.: • relatively fast
  • don't need derivative explicity

– otherwise similar to Newton

• Python's brentq fct.:
  – combines advantages of several methods (especially bisection and secant)
  – always converges for cont. fct.s
$\Rightarrow$ robust and relatively fast