2. Options and Binomial Tree Models

Session 6
Sep. 21, 2018

2.1 Options Basics

option = contract depending on future price of some underlying asset
(e.g., stock, usually assumed here)

=> this is a type of "derivative" financial instrument

call option: holder can buy underlying asset for price K at time T

put option: holder can sell underlying asset for price K at time T

option characterized by:
- expiration date T
- strike price K

terminology:
- payoff = value of option at expiration T
- underlying asset's price S

Ex.: strike price 50 \$

suppose at T the stock price 60 \$

↳ call option (buy): payoff = 10 \$ (exercise option)

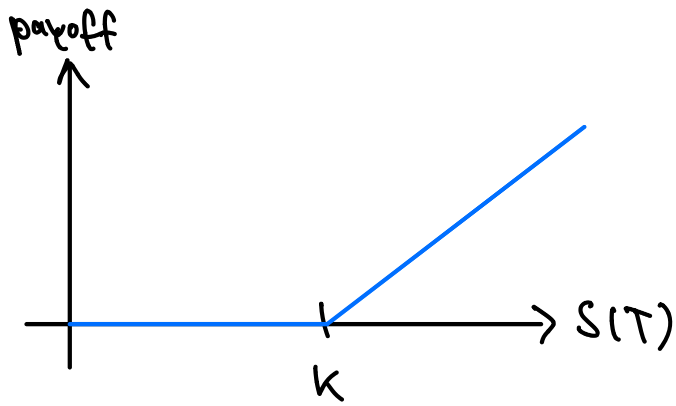
↳ put option (sell): payoff = 0 \$ (not exercise option) note: profit = payoff − option price

European option: can be exercised only at expiration

American option: can be exercised any time at or before expiration

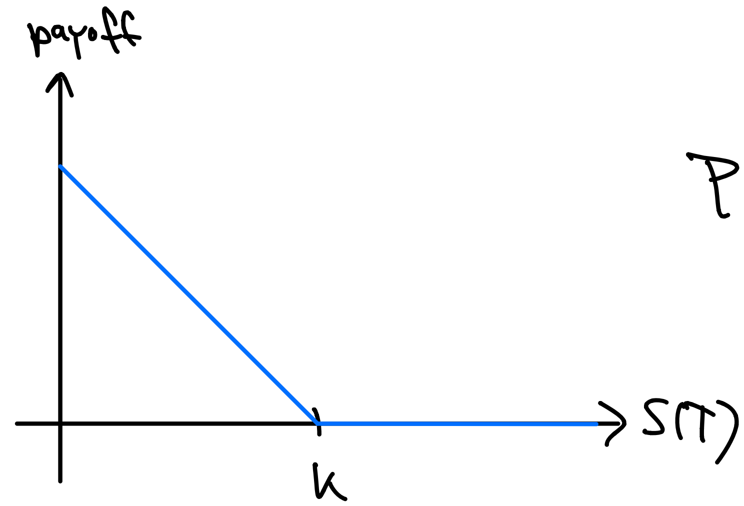
options can be used, e.g., as insurance, speculation etc.

call payoff:



$C = \max(0, S(T) - K)$

put payoff:

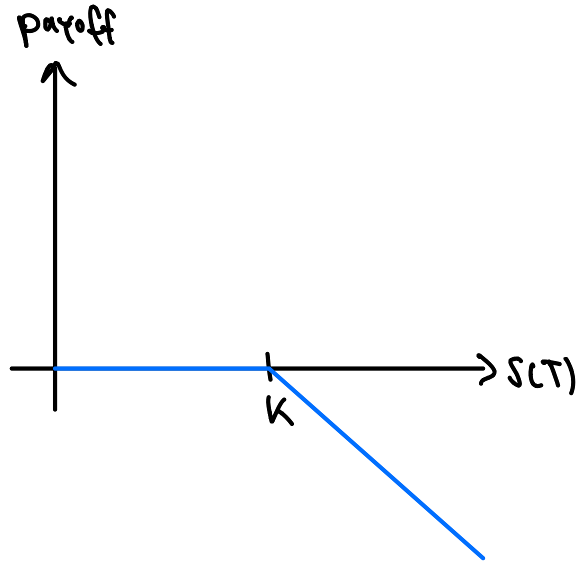


$P = \max(0, K - S(T))$

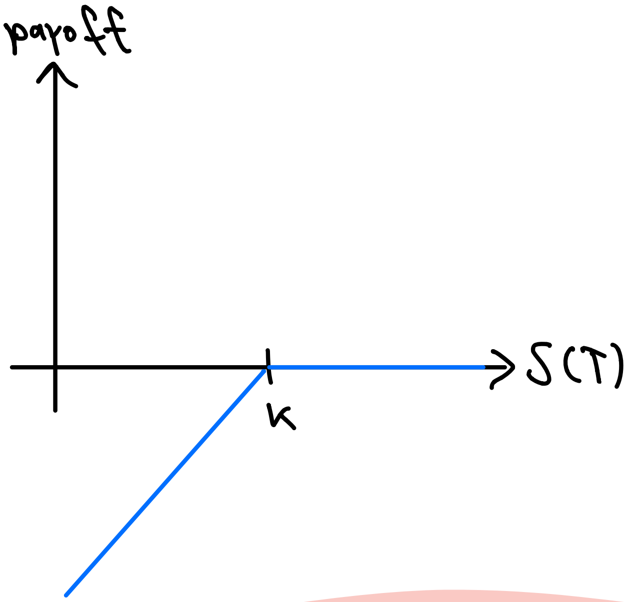
buying option: long position (we usually assume that)

selling option: short position

↳ short a call:



↳ short a put:



<u>Goal</u> (for most of the rest of this class):

What should be price of an option?

<u>Assumptions:</u>

- there is a risk-free market, which we take to be the bond market, with risk-free interest rate r (e.g., US treasury bonds)
- stocks / bonds can be bought and sold unlimitedly and without transaction costs

Uncertainty: $S(T) \longrightarrow$ some probabilistic model

starting point for all pricing models:

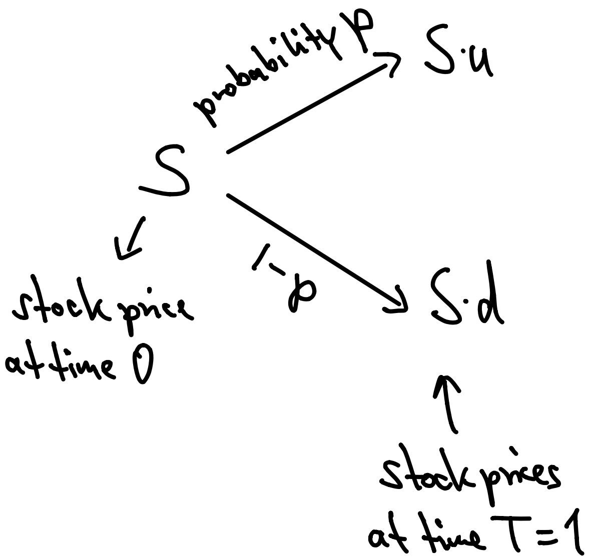
no opportunity for risk-free profit

= no arbitrage assumption

2.2 Binary Model

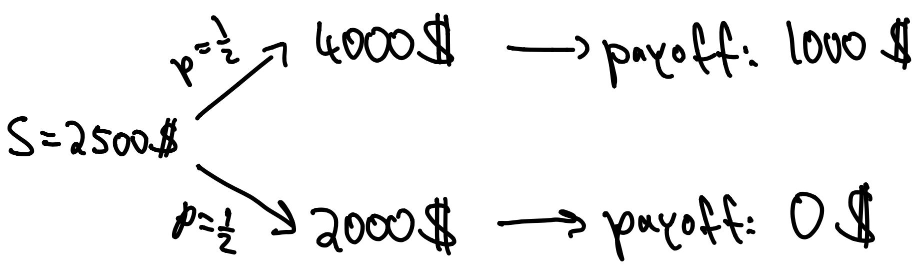
first we ask this question within a simple binary model

$\longrightarrow$ 2 possibilities for $S(T)$



$d < u$

Example: $S = 2500\$$, $K = 3000\$$, $r = 0$, call (long)



one possibility: set price at $C = \frac{1}{2} \cdot 1000\$ + \frac{1}{2} \cdot 0\$ = 500\$$

then the seller cold have the following strategy:

sell option, borrow 2000\$, buy one stock (for 2500\$)

↳ if $S(T) = 4000\$$ → option will be exercised, sell stock for $K = 3000\$$

⇒ profit: $3000\$ - 2000\$ = 1000\$$

↳ if $S(T) = 2000\$$ → option will not be exercised, sell stock for 2000\$

⇒ profit: $2000\$ - 2000\$ = 0\$$

⇒ bad option price, since seller has opportunity for risk-free profit

general idea: construct portfolio that mimics option price, called

replicating portfolio

x_1 = price of bond with riskless interest rate r, continuously compounded

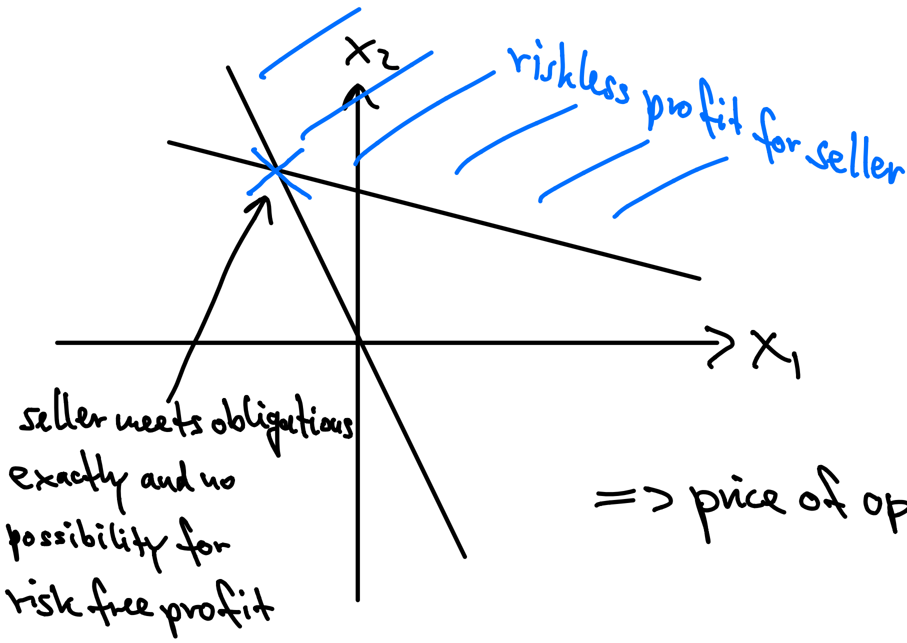
(we assume $d < e^r < u$)

x_2 = # of stocks at price S = "hedge ratio" or "delta"

in general: riskless or zero profit for seller if two conditions hold:

$$e^r x_1 + Su \cdot x_2 \geq C_u \qquad (C_u = \text{payoff at } Su)$$

$$\underbrace{e^r x_1 + Sd\, x_2}_{\text{value seller has at } T} \geq \underbrace{C_d}_{\text{what seller needs to pay}} \qquad (C_d = \text{payoff at } Sd)$$



=> price of option = cost of portfolio at this intersection

=> set up replicating portfolio here

=> price $C = x_1 + S x_2$ with x_1 and x_2 determined by

$e^r x_1 + Su\, x_2 = C_u$ and $e^r x_1 + S \cdot d \cdot x_2 = C_d$

<u>Ex. from above:</u> replicating portfolio

$$x_1 + 4000 x_2 = 1000$$
$$x_1 + 2000 x_2 = 0$$

$\Rightarrow 2000 x_2 = 1000 \Rightarrow x_2 = \frac{1}{2} \Rightarrow x_1 = -1000$

portfolio costs $-1000\$ + \frac{1}{2} \cdot 2500\$ = 250\$ = C$ fair price,

since no possibility for risk-free profit

$(C = x_1 + S x_2$ for x_1, x_2 above$)$

Ex. again:

	Seller: borrow $1000\$$, $\overbrace{\text{buy } \tfrac{1}{2} \text{ stock}}^{\text{for } 1250\$}$	buyer: buys option for $250\$$
$S_T = 4000\$$	buy $\frac{1}{2}$ stock: $2000\$$ sell stock for $3000\$$ $\Rightarrow$ profit: $-1000\$ - 2000\$ + 3000\$ = 0\$$ alternative: buy 2 stocks profit: $250\$ - 5000\$ + 3000\$$ $+ 4000\$ = 2250\$$	buy stock at $K = 3000\$$ $\Rightarrow$ profit: $\underbrace{1000\$}_{4000\$ - 3000\$} - 250\$ = 750\$$
$S_T = 2000\$$	sell $\frac{1}{2}$ stock: $1000\$$ $\Rightarrow$ profit: $-1000\$ + 1000\$ = 0\$$ alternative: buy 2 stocks profit: $250\$ - 5000\$ + 2000\$$ $+ 2000\$ = -750\$$	not exercise option $\Rightarrow$ profit: $-250\$$

note: • one can actually buy $\frac{1}{2}$ stock

↳ called "fractional share" (used, i.e., for dividend reinvestment)