Session 8
Sep. 28, 2018

replicating portfolio

x_1 = price of bond with riskless interest rate r,

continuous compounding (we assume $d < e^r < u$)

x_2 = # of stocks at price S = "hedge ratio"

no arbitrage $\Rightarrow$ $C = x_1 + S x_2$ with
$$e^r x_1 + S u x_2 = C_u$$
$$e^r x_1 + S d x_2 = C_d$$
(strike price X)

with payoffs $C_u = \max(0, Su - X)$ for call options
$C_d = \max(0, Sd - X)$

solve this for x_1, x_2

$$\Rightarrow \quad Su x_2 - Sd x_2 = C_u - C_d \quad \Rightarrow \quad x_2 = \frac{C_u - C_d}{Su - Sd}$$

$$\Rightarrow \quad x_1 = e^{-r}(C_d - Sd x_2)$$

$$= e^{-r}\left(C_d - \frac{Sd(C_u - C_d)}{Su - Sd}\right)$$

$$= e^{-r}\left(\frac{(u-d)C_d - d(C_u - C_d)}{u-d}\right) = e^{-r}\left(\frac{uC_d - dC_u}{u-d}\right)$$

$$\Rightarrow C = x_1 + S x_2 = e^{-r}\left(\frac{uC_d - dC_u}{u-d}\right) + S\left(\frac{C_u - C_d}{Su - Sd}\right)$$

$$= e^{-r}\left(C_d \underbrace{\frac{u-e^r}{u-d}}_{p_d} + C_u \underbrace{\frac{e^r-d}{u-d}}_{p_u}\right)$$

$$\Rightarrow C = e^{-r}\left(p_d C_d + p_u C_u\right)$$

note: • $p_d = \frac{u-e^r}{u-d} = \frac{u-d+d-e^r}{u-d} = 1-p_u$

• we assumed $d < e^r < u \Rightarrow 0 < p_d < 1$ and $0 < p_u < 1$

$\Rightarrow p_u, p_d$ are called risk-neutral probabilities

What would be the expectation value of stock price at T under probabilities p_u, p_d?

$$\Rightarrow \mathbb{E}\left(S(T)_{p_u, p_d}\right) = p_u Su + p_d Sd$$

$$= \left(\frac{e^r-d}{u-d}\right) Su + \left(\frac{u-e^r}{u-d}\right) Sd$$

$$= S\left(\frac{(e^r-d)u + (u-e^r)d}{u-d}\right)$$

$= e^r S$, i.e., expected rate of return = riskless rate r (under risk-neutral probabilities)

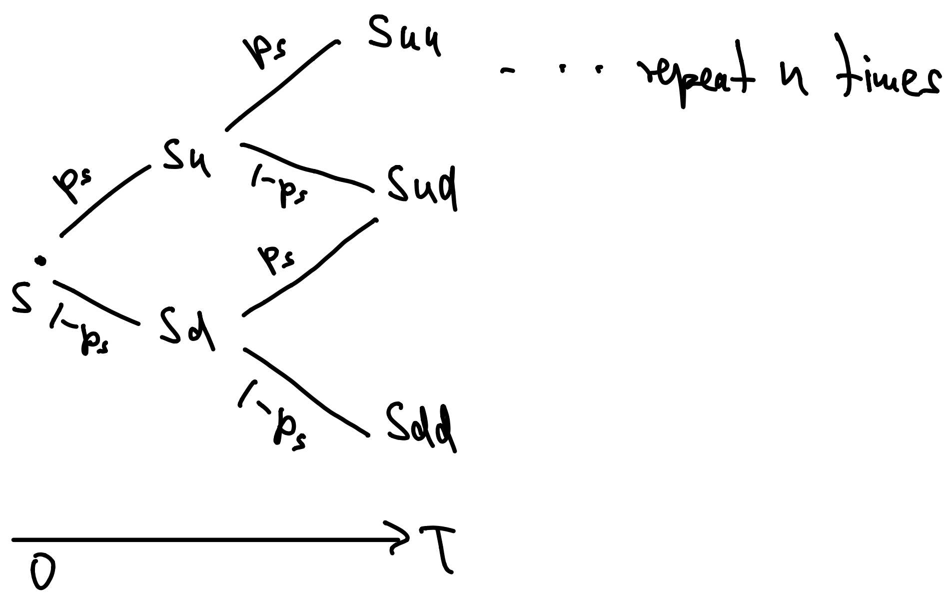
- remarkable: here result C is independent of probabilities of stock price model (10% chance going up, 90% down $\Rightarrow$ same price)

2.3 Binomial Tree Models

we just repeat the binary model with many steps

$\longrightarrow$ binomial tree

<u>Model for stock price development:</u>

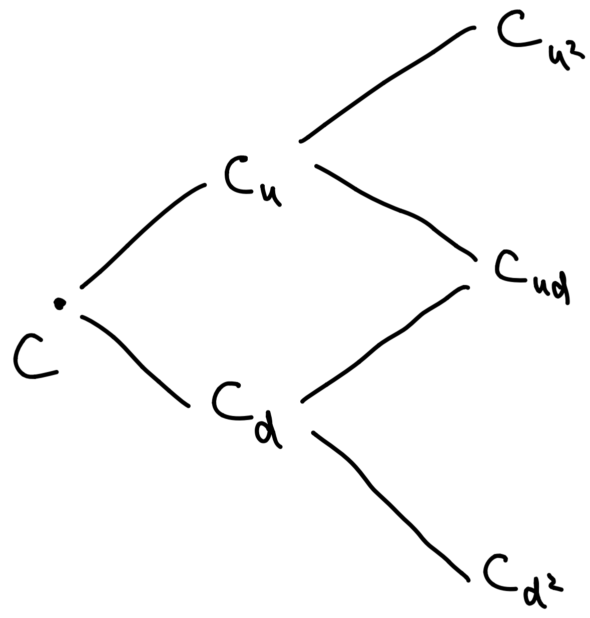


$\Rightarrow$ stock price $S_T^{j\,up} = S u^j d^{n-j}$ (n steps)

probability $P(j,n) = \underbrace{\binom{n}{j}}_{\frac{n!}{(n-j)!\,j!}} p_s^j (1-p_s)^{n-j}$ $\left(\text{remember } (a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}\right)$

length of period $= \frac{T}{n}$

Option Price:



do it explicitly for calls here

for $n = 2$: start at last step (last column), at time of expiration

given: $C_{u^2} = \max(0, Su^2 - X)$, X = strike price

$C_{ud} = \max(0, Sud - X)$

$C_{d^2} = \max(0, Sd^2 - X)$

$C_u = e^{-r}\left(p\,C_{u^2} + (1-p)\,C_{ud} \right)$ where $p = p_u \Rightarrow p_d = 1 - p_u = 1 - p$

$C_d = e^{-r}\left(p\,C_{ud} + (1-p)\,C_{d^2} \right)$

last step: $C = e^{-r}\left(p\,C_u + (1-p)\,C_d \right)$

$= e^{-2r}\left(p^2 C_{u^2} + 2p(1-p)\,C_{ud} + (1-p)^2 C_{d^2} \right)$

in this case we get an explicit formula for the option price:

for n periods: $$C = e^{-nr} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \max(0, S u^j d^{n-j} - X)$$

(note: in terms of the period interest rate r_p we should use $r = r_p \frac{T}{n}$)

In the general case or with more complicated models (e.g., dividend payments or discontinuous interest compounding) there might not be closed-form formulas => better to implement bin. tree by "backward induction"

note: bin. tree model is very versatile (complicated models can be implemented in a simple way)

python implementation:

- to store data: – vectors (memory efficient)
  – matrix (if you need all data, e.g., for plots)

- for going from one column to previous one use vectorized operations
  ↳ use only one "for" loop to go through all steps

to find suitable u, d we need to "calibrate" model

The stock's rate of return is $S_T^{j\,up} = e^{Y_j} S$ for j upward movements

$$\Rightarrow Y_j = \ln\left(\frac{S_T^{j\,up}}{S}\right) = \ln \frac{S u^j d^{n-j}}{S} = \ln(u^j d^{n-j})$$

calibrate or choose u, d such that

- expectation value $\mathbb{E}(Y_j) \longrightarrow \mu \cdot T$ as $n \to \infty$
- variance $Var(Y_j) \longrightarrow \sigma^2 \cdot T$ as $n \to \infty$

μ = mean growth of stock , σ = volatility of stock (standard deviation)

we look at that next time ...

for HW problem 2 , take $u = \frac{1}{d} = e^{\sigma\sqrt{\frac{T}{n}}}$