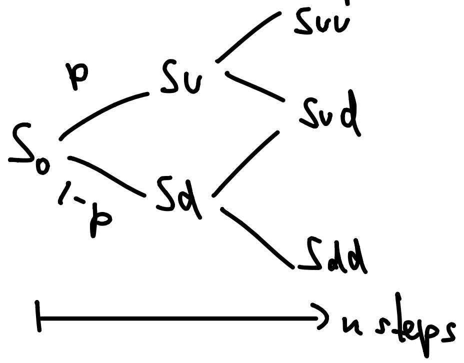
2.4 Binomial Tree and Calibration

Session 10
Oct. 5, 2018

recall: • model for stock price development



recall: $\sum_{j=0}^{n} P(j,n) = \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j}$
$= (p+(1-p))^n = 1$

probability for j up's $=: P(j,n) = \binom{n}{j} p^j (1-p)^{n-j}$

• stock's rate of return:

$\ln(ab) = \ln(a) + \ln(b)$, $\ln x^a = a \ln x$

$$Y_j = \ln \frac{S_T^{j\,up}}{S_0} = \ln u^j d^{n-j} = \ln\left(\left(\frac{u}{d}\right)^j d^n\right) = j \ln\left(\frac{u}{d}\right) + n \ln d$$

$$\left(S_T^{j\,up} = S_0 e^{Y_j} \right)$$

next we want to compute expectation and variance of Y ($Y = Y_j$ fct. of j)

<u>Def.</u>: • Expectation value of X is $\mathbb{E}(X) = \sum_{j=0}^{n} X_j P(j,n)$

• Variance of X is $Var(X) = \mathbb{E}\left((X - \mathbb{E}(X))^2 \right)$

Calculation rules:

• $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$, $\mathbb{E}(\lambda X) = \lambda \mathbb{E}(X)$ $(\lambda \in \mathbb{R})$

- $\mathrm{Var}(X) = \mathbb{E}\left((X-\mathbb{E}(X))^2\right) = \mathbb{E}\left(X^2 - 2X\,\mathbb{E}(X) + \mathbb{E}(X)^2\right)$

$$= \mathbb{E}(X^2) + \underbrace{\mathbb{E}(-2X\,\mathbb{E}(X))}_{=-2\,\mathbb{E}(X)^2} + \mathbb{E}(X)^2$$

$$= \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

- $\mathrm{Var}(\lambda X) = \lambda^2\,\mathrm{Var}(X)$

- $\mathrm{Var}(X+Y) = \mathbb{E}((X+Y)^2) - \mathbb{E}(X+Y)^2$

$$= \mathbb{E}(X^2+2XY+Y^2) - \mathbb{E}(X)^2 - 2\,\mathbb{E}(X)\mathbb{E}(Y) - \mathbb{E}(Y)^2$$

$$= \underbrace{\mathbb{E}(X^2)-\mathbb{E}(X)^2}_{\mathrm{Var}(X)} + \underbrace{\mathbb{E}(Y^2)-\mathbb{E}(Y)^2}_{\mathrm{Var}(Y)} + \underbrace{2\left(\mathbb{E}(XY) - \mathbb{E}(X)\cdot\mathbb{E}(Y)\right)}_{=\mathrm{Cov}(X,Y)}$$

$= \mathrm{Cov}(X,Y)$ (covariance of X and Y)

$\mathrm{Cov}(X,Y)=0$ if X and Y are independent

next: compute $\mathbb{E}(j)$, $\mathbb{E}(j^2)$, which means:

let $f(j)=j$, $g(j)=j^2$, then we compute $\mathbb{E}(f)$ and $\mathbb{E}(g)$

"loose notation": $\mathbb{E}(f) \equiv \mathbb{E}(j)$, $\mathbb{E}(g) \equiv \mathbb{E}(j^2)$

- $\mathbb{E}(j) = \sum_{j=0}^{n} j\,P(j,n) = \sum_{j=0}^{n} j \binom{n}{j} p^j (1-p)^{n-j} = \sum_{j=0}^{n} j \binom{n}{j} \left(\frac{p}{1-p}\right)^j (1-p)^n$

$$\sum_{j=0}^{n} j \binom{n}{j} x^j = \sum_{j=1}^{n} j \frac{n!}{(n-j)!\,j!} x^j = \sum_{j=1}^{n} \frac{n!}{(n-j)!\,(j-1)!} x^j$$

$$= \sum_{j=1}^{n} n \underbrace{\frac{(n-1)!}{(n-1-(j-1))!\,(j-1)!}}_{=\binom{n-1}{j-1}} x^j$$

$$= n \sum_{j=1}^{n} \binom{n-1}{j-1} x^j$$

$$= nx \underbrace{\sum_{j=1}^{n} \binom{n-1}{j-1} x^{j-1}}_{= \sum_{j=0}^{n-1} \binom{n-1}{j} x^j = (1+x)^{n-1}}$$

$$= nx\,(1+x)^{n-1}$$

alternatively: $$\sum_{j=0}^{n} j \binom{n}{j} x^j = x \overbrace{\frac{d}{dx}\left(\sum_{j=0}^{n} \binom{n}{j} x^j \right)}^{\sum_{j=0}^{n} \binom{n}{j} j x^{j-1}}$$

$$= x \frac{d}{dx} (1+x)^n$$

$$= x\, n\, (1+x)^{n-1}$$

$$\Rightarrow \mathbb{E}(j) = n \underbrace{\left(\frac{p}{1-p}\right)}_{x} \Big(1 + \underbrace{\frac{p}{1-p}}_{x}\Big)^{n-1} (1-p)^n$$

$$= n \frac{p}{1-p} \left(\frac{1}{1-p}\right)^{n-1} (1-p)^n$$

$$= n \cdot p$$

- by similar computation: $\mathbb{E}(j^2) = np\big((n-1)p + 1\big)$

$$\Rightarrow \mathrm{Var}(j) = \mathbb{E}(j^2) - \mathbb{E}(j)^2 = np(1-p)$$

then we find: (recall $Y_j = j\ln\left(\frac{u}{d}\right) + n\ln d$)

- $\mathbb{E}(Y_j) = \mathbb{E}(j)\cdot \ln\frac{u}{d} + n\ln d = n\cdot p\ln\frac{u}{d} + n\ln d$

- $\mathrm{Var}(Y_j) = \mathrm{Var}\left(j\ln\frac{u}{d} + n\ln d\right) = \left(\ln\frac{u}{d}\right)^2 \mathrm{Var}(j) + \underbrace{\mathrm{Var}(n\ln d)}$

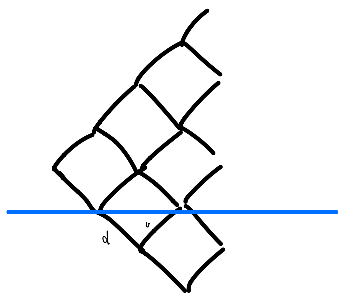
$$= np(1-p)\left(\ln\frac{u}{d}\right)^2$$

now: calibrate our model, meaning that we want

$$\mathbb{E}(Y_j) \xrightarrow{n\to\infty} \mu T \quad , \quad \mathrm{Var}(Y_j) \xrightarrow{n\to\infty} \sigma^2 T$$

μ = mean value $\qquad$ σ = volatility

one sensible condition is $u\cdot d = 1$



same value on each horizontal line

$$\Rightarrow \mathbb{E}(Y_j) = 2np\ln u - n\ln u$$
$$= (\ln u)\, n\,(2p-1)$$

$$\mathrm{Var}(Y_j) = 4(\ln u)^2 np(1-p)$$

one possible choice: $p = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{T}{n}}$

$$u = e^{\sigma\sqrt{\frac{T}{n}}}$$

$$\Rightarrow \mathbb{E}(Y_j) = \ln e^{\sigma\sqrt{\frac{T}{n}}}\, n \left(1 + \frac{\mu}{\sigma}\sqrt{\frac{T}{n}} - 1 \right)$$

$$= n\,\sigma\sqrt{\frac{T}{n}}\,\frac{\mu}{\sigma}\sqrt{\frac{T}{n}} = \mu T$$

$$\Rightarrow Var(Y_j) = 4\left(\ln e^{\sigma\sqrt{\frac{T}{n}}}\right)^2 n \,\frac{1}{4}\left(1 + \frac{\mu}{\sigma}\sqrt{\frac{T}{n}}\right)\left(1 - \frac{\mu}{\sigma}\sqrt{\frac{T}{n}}\right)$$

$$= 4\left(\sigma\sqrt{\frac{T}{n}}\right)^2 n\,\frac{1}{4}\left(1 - \frac{\mu^2}{\sigma^2}\frac{T}{n}\right)$$

$$= \sigma^2 T \left(1 - \underbrace{\frac{\mu^2}{\sigma^2}\frac{T}{n}}_{\xrightarrow{n\to\infty} 0}\right) \xrightarrow{n\to\infty} \sigma^2 T$$

note: another possibility is $p = \frac{1}{2}$

$$u = \exp\left(\mu\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}\right)$$

$$d = \exp\left(\mu\frac{T}{n} - \sigma\sqrt{\frac{T}{n}}\right)$$

($u \cdot d \neq 1$ here)

2.5 Convergence Rates

Next week we want to compute limit $n\to\infty$ for option pricing, i.e.,

we compute $\lim_{n\to\infty} C_n(T=0) = C(T=0)$. This $C(T=0)$ will be

given by Black-Scholes formula.

How fast is convergence?

usually $|C_n - C| \approx A\cdot n^{-\beta}$, A = some constant

β = rate of convergence

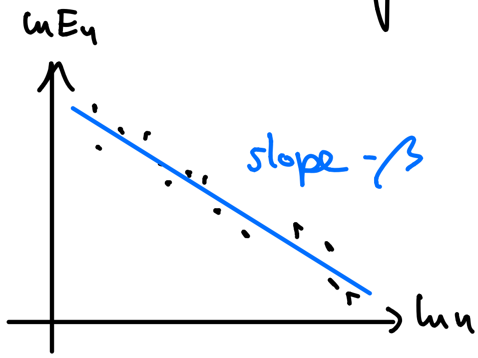
note: if C is unknown, we could look at $|C_n - C_N|$ for some $N >> n$

Here, we read off convergence rates from plots.

We plot $E_n = |C_n - C|$ against n

better: $\ln E_n \approx \ln A n^{-\beta} = \ln A - \beta \ln n$

If we plot $\ln E_n$ against n we (hopefully) get a line with slope $-\beta$



python: loglog$(n, E_n) \Leftrightarrow$ plot$(\ln n, \ln E_n)$

$\hookrightarrow$ use this for Problem 3 (HW5)