2.6 Central Limit Theorem

Session 12
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Binomial distribution:

"up" with probability p, "down" with probability $1-p$

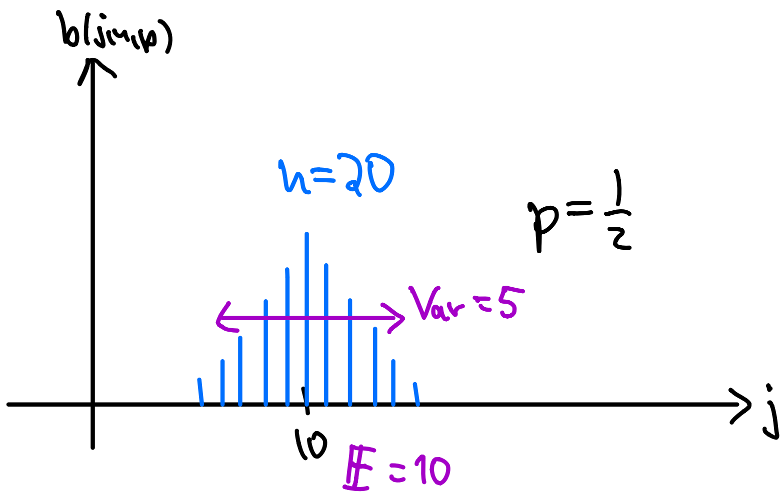
$b(j,n,p)$ = probability of j up's in n trials

$$b(j,n,p) = \binom{n}{j} p^j (1-p)^{n-j} \quad , \binom{n}{j} = \frac{n!}{(n-j)!\,j!}$$

note/recall: • $\sum_{j=0}^{n} b(j,n,p) = \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} = (p+(1-p))^n = 1$

• $\mathbb{E}(j) = np$

• $\mathrm{Var}(j) = np(1-p)$



center the distribution by shifting $Y_j = j - \mathbb{E}(j) = j - np$

$\Rightarrow \mathbb{E}(Y_j) = \mathbb{E}(j) - np = np - np = 0$

normalize variance by setting $X_j = \frac{j-np}{\sqrt{np(1-p)}}$

$\mathrm{Var}(\lambda x) = \lambda^2 \mathrm{Var}(x)$

$\Rightarrow \mathbb{E}(X_j) = 0$ and $\mathrm{Var}(X_j) = \frac{1}{np(1-p)} \mathrm{Var}(j-np) = 1$

$$\sum_{j=0}^{n} b(j,n,p) = 1$$

$$j = \sqrt{np(1-p)}\,x + np \quad \Big| \quad dj = \sqrt{np(1-p)}\,dx$$

in the limit $n\to\infty$ we would expect $\sum_j \Delta j \xrightarrow{n\to\infty} \int dj = \int \sqrt{np(1-p)}\,dx$

Central Limit Theorem for binomial distribution:

$$\sqrt{np(1-p)}\; b\left(\sqrt{np(1-p)}\,x + np, n, p\right) \xrightarrow{n\to\infty} \varphi(x)$$, where $\varphi(x)$ is the Gaussian with mean 0 and variance 1, i.e., $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} =: \mathcal{N}(\underset{\text{mean}}{0}, \underset{\text{variance}}{1})$

note: • $$\left(\int_{-\infty}^{\infty} \varphi(x)dx\right)^2 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \frac{1}{2\pi} e^{\frac{-x^2-y^2}{2}} = \int_0^{\infty} r\,dr \int_0^{2\pi} d\varphi\, e^{-\frac{r^2}{2}} \frac{1}{2\pi}$$

$$= \int_0^{\infty} dr\, r e^{-\frac{r^2}{2}} = -e^{-\frac{r^2}{2}}\Big|_0^{\infty} = 1$$

so it is indeed normalized

• check: $$\mathbb{E}(x) = \int_{-\infty}^{\infty} x\varphi(x)dx = \underbrace{\int_{-\infty}^{0} x\varphi(x)dx}_{} + \int_0^{\infty} x\varphi(x)dx = 0$$

$$= -\int_0^{-\infty} x\varphi(x)dx$$

$$\overset{x\to -x}{=} -\int_0^{\infty} (-x)\varphi(-x)(-dx)$$

$$= -\int_0^{\infty} x\varphi(x)dx$$

(x odd fct., $\varphi(x)$ even ($\varphi(-x) = \varphi(x)$))

• check: $\mathrm{Var}(X) = \int_{-\infty}^{\infty} x^2 \varphi(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}}\,dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \left(x e^{-\frac{x^2}{2}}\right) dx$$

integration by parts:
$\int fg = fG - \int f'G$

$$= \frac{1}{\sqrt{2\pi}} \left(x\left(-e^{-\frac{x^2}{2}}\right)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \left(-e^{-\frac{x^2}{2}}\right) dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx$$

$$= 1$$

Proof of Thm.:

$$j = \sqrt{np(1-p)}\,x + np = n(p + \alpha x), \quad \alpha = \sqrt{\frac{p(1-p)}{n}}$$

$$\sqrt{np(1-p)}\, b(j, n, p) = \sqrt{np(1-p)} \underbrace{\binom{n}{n(p+\alpha x)}} p^{n(p+\alpha x)} (1-p)^{n(1-p-\alpha x)}$$

$$= \frac{n!}{[n(1-p-\alpha x)]!\,[n(p+\alpha x)]!}$$

Stirling approximation: $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ (see HW)

$$\ln n! = \sum_{i=1}^{n} \ln i \approx \int_1^n \ln x\,dx = \int_1^n 1 \cdot \ln x\,dx = \ln x \cdot x \Big|_1^n - \int_1^n x \frac{1}{x}\,dx$$

$\ln ab = \ln a + \ln b$

$$= n \ln n - \int_1^n dx = n \ln n - n + 1 \approx n \ln n - n$$

$$\Rightarrow n! \approx e^{n \ln n - n} = n^n e^{-n} = \left(\frac{n}{e}\right)^n$$

$$\Rightarrow \sqrt{np(1-p)}\, b(j;n,p)$$

$$\approx \sqrt{np(1-p)}\; \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n p^{n(p+\alpha x)} (1-p)^{n(1-p-\alpha x)}}{\sqrt{2\pi n(1-p-\alpha x)}\left(\frac{n(1-p-\alpha x)}{e}\right)^{n(1-p-\alpha x)} \sqrt{2\pi n(p+\alpha x)}\left(\frac{n(p+\alpha x)}{e}\right)^{n(p+\alpha x)}}$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{p(1-p)}{(p+\alpha x)(1-p-\alpha x)}}\; \underbrace{\frac{\left(\frac{n}{e}\right)^n}{\left(\frac{n}{e}\right)^{n(1-p-\alpha x)}\left(\frac{n}{e}\right)^{n(p+\alpha x)}}}_{=1} \left(\frac{p}{p+\alpha x}\right)^{n(p+\alpha x)} \left(\frac{1-p}{1-p-\alpha x}\right)^{n(1-p-\alpha x)}$$

note: $\cdot\ \frac{p}{p+\alpha x} = \frac{p+\alpha x-\alpha x}{p+\alpha x} = 1 - \frac{\alpha x}{p+\alpha x}$

Taylor series: $(1-\gamma)^b = 1 + \gamma\left[(1-\gamma)^b\right]'_{\gamma=0} + \frac{\gamma^2}{2}\left[(1-\gamma)^b\right]''_{\gamma=0} + O(\gamma^3)$

$$= 1 - b\gamma + \frac{b(b-1)}{2}\gamma^2 + O(\gamma^3)$$

$$\Rightarrow \left(1-\frac{\alpha x}{p+\alpha x}\right)^{p+\alpha x} = 1 - \frac{(p+\alpha x)\alpha x}{p+\alpha x} + \frac{(p+\alpha x)(p+\alpha x-1)\,\alpha^2 x^2}{2(p+\alpha x)^2} + O(\alpha^3)$$

$$= 1 - \alpha x - \frac{(1-p)}{2p}\alpha^2 x^2 + O(\alpha^3)$$

by replacing $p \to 1-p$, $\alpha \to -\alpha$:

$$\Rightarrow \left(\frac{1-p}{1-p-\alpha x}\right)^{1-p-\alpha x} = 1 + \alpha x - \frac{p}{2(1-p)}\alpha^2 x^2 + O(\alpha^3)$$

$$\Rightarrow \left(\frac{p}{p+\alpha x}\right)^{p+\alpha x} \left(\frac{1-p}{1-p-\alpha x}\right)^{1-p-\alpha x}$$

$$= \left(1-\alpha x - \frac{(1-p)}{2p}\alpha^2 x^2 + \mathcal{O}(\alpha^3)\right)\left(1+\alpha x - \frac{p}{2(1-p)}\alpha^2 x^2 + \mathcal{O}(\alpha^3)\right)$$

$$= 1 + \alpha x - \frac{p}{2(1-p)}\alpha^2 x^2 - \alpha x - \alpha^2 x^2 - \frac{(1-p)}{2p}\alpha^2 x^2 + \mathcal{O}(\alpha^3)$$

$$= 1 - \underbrace{\left(\frac{p}{2(1-p)} + 1 + \frac{(1-p)}{2p}\right)}\alpha^2 x^2 + \mathcal{O}(\alpha^3)$$

$$= \frac{p\cdot p + 2(1-p)p + (1-p)^2}{2(1-p)p} = \frac{p^2 + 2p - 2p^2 + 1 - 2p + p^2}{2(1-p)p} = \frac{1}{2(1-p)p}$$

$$\Rightarrow \sqrt{np(1-p)}\, b(j;n,p) = \frac{1}{\sqrt{2\pi}} \underbrace{\sqrt{\frac{p(1-p)}{(p+\alpha x)(1-p-\alpha x)}}}_{\to 1 \text{ for } n\to\infty} \underbrace{\left(1 - \frac{\alpha^2 x^2}{2(1-p)p} + \mathcal{O}(\alpha^3)\right)^n}$$

$$= \left(1 - \frac{x^2}{2n} + \mathcal{O}(n^{-\frac{3}{2}})\right)^n \xrightarrow{n\to\infty} e^{-\frac{x^2}{2}}$$

$$\Rightarrow \sqrt{np(1-p)}\, b(j;n,p) \xrightarrow{n\to\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad \square$$

2.7 Black-Scholes Formula

recall: option price for European calls:

$$C = e^{-rT} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \max(0, S u^j d^{n-j} - K)$$

$$= e^{-rT}\, \mathbb{E}(\text{payoff}) \qquad (r = \text{period interest rate},\ K = \text{strike price})$$

when is payoff $\neq 0$, i.e., $Su^j d^{n-j} - K > 0$?

$$\Rightarrow \dots \quad j > \frac{\ln \frac{K}{Sd^n}}{\ln \frac{u}{d}} =: a$$

$$\Rightarrow C = e^{-rT} \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \left(Su^j d^{n-j} - K \right)$$

$$= S \sum_{j=a}^{n} \binom{n}{j} \left(pue^{-r\frac{T}{n}}\right)^j \left((1-p)de^{-r\frac{T}{n}}\right)^{n-j} - Ke^{-rT} \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j}$$

recall: $p = \dfrac{e^{r\frac{T}{n}} - d}{u - d}$

this gives $(1-p)de^{-r\frac{T}{n}} = \dots = 1 - pue^{-r\frac{T}{n}}$

$$\Rightarrow C = S \sum_{j=a}^{n} b(j, n, pue^{-r\frac{T}{n}}) - Ke^{-rT} \sum_{j=a}^{n} b(j, n, p)$$

next: use calibration $u = e^{\sigma\sqrt{\frac{T}{n}}}$, $d = \frac{1}{u}$

$\Rightarrow$ compute p and a and take limit $n \to \infty$, i.e., use central limit thm.

$\Rightarrow$ lengthy computation

just note: $\displaystyle \lim_{n \to \infty} \frac{a - np}{\sqrt{np(1-p)}} = \frac{\ln \frac{K}{S} - T\left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T}}$

$$p = \frac{1}{2}\left(1 + \frac{\left(r + \frac{\sigma^2}{2}\right)}{\sigma}\sqrt{\frac{T}{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right)$$

Result: $C = S\,\Phi(x) - Ke^{-rT}\,\Phi(x - \sigma\sqrt{T})$

with $x = \dfrac{\ln\frac{S}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$,

where $\Phi(x) = \int_{-\infty}^{x} \varphi(y)\,dy = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,dy$

(cumulative normal distribution fct.)

This is the Black-Scholes Formula.